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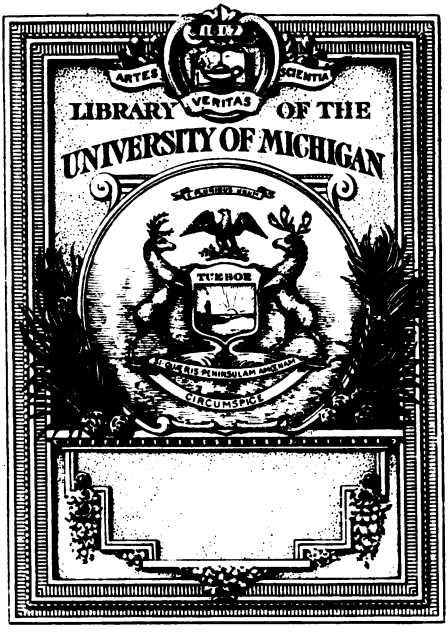
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THE

PRACTICAL ANALYST;

OR,

A TREATISE ON ALGEBRA,

CONTAINING THE MOST USEFUL PARTS OF THAT SCIENCE,

Illustrated by a Copious Collection of Examples,

DESIGNED FOR THE USE OF SCHOOLS.

BY ENOCH LEWIS.

PHILADELPHIA:

PUBLISHED BY KIMBER AND SHARPLESS,

No. 93 MARKET STREET.

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1826.

Eastern District of Pennsylvania, to wit:

SEAL. BE IT REMEMBERED, that on the seventeenth day of August, in the fifty-first year of the independence of the United States of America, A. D. 1826, Kimber and Sharpless, of the said district, have deposited in this office the title of a book, the right whereof they claim as proprietors, in the words following, to wit:

"The Practical Analyst; or, a Treatise on Algebra, containing the most useful parts of that Science, illustrated by a copious Collection of Examples, designed for the Use of Schools, by Enoch Lewis."

In conformity to the Act of the Congress of the United States, intituled, "An Act for the Encouragement of Learning, by securing the Copies of Maps, Charts, and Books, to the Authors and proprietors of such Copies, during the times therein mentioned."—And also to the Act, entitled, "An Act supplementary to an Act, entitled, 'An Act for the Encouragement of Learning, by securing the Copies of Maps, Charts, and Books, to the Authors and Proprietors of such Copies during the times therein mentioned,' and extending the benefits thereof to the arts of designing, engraving, and etching Historical and other Prints."

D. CALDWELL,
Clerk of the Eastern District of Pennsylvania.

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PREFACE.

THE mathematical sciences, have been considered, since the early periods of their existence, among the noblest objects of human inquiry. Of these, next to arithmetic, geometry occupies the first place, in regard both to importance and to time. The Grecian philosophers cultivated this branch of knowledge, with an ardour and industry, which manifest their high opinion of its value. In the Ionian and Pythagorean schools, geometry was considered an indispensable preliminary to the study of philosophy. When a person, ignorant of geometry, applied for his instructions to the philosopher Xenocrates, he is said to have made this laconic reply: "thou hast not the handles of philosophy."

The perspicuity of geometrical reasoning, the accurate and inimitable dependence of its arguments, and the unfaltering certainty of its conclusions, are eminently calculated to form the mind to habits of attention, and to a regular and forcible concatenation of its ideas. No wonder then, that it was so highly prized by that acute and inquisitive people.

Two modes of procedure, according to the different objects in view, were adopted by the ancient geometers: synthesis and analysis.

Synthesis, or composition, consists in the direct solution of a problem; or, the demonstration of a proposition by a series of arguments, regularly deduced, from self-evident truths, or from other propositions previously established. This method is very proper for conveying,

with clearness, to another, those truths which are completely understood by the instructer. The works of the ancient geometers, which have escaped the ravages of the barbarous ages, are mostly synthetical. The Elements of Euclid, so well known in our modern schools of geometry, furnish, probably, the most complete specimen of the ancient synthesis extant. And it is a remarkable fact in the history of science, that the geometry of Euclid, though written two thousand years ago, and some time anterior to any other mathematical tract that has reached us, is still one of the best elementary works on that subject, which has ever appeared.

Analysis, or decomposition, on the other hand, assumes, as known, the proposition which is to be examined; or, as already effected the solution which is to be made, and thence proceeds to examine the consequences necessarily resulting from such supposition, until, in case of a theorem, a conclusion is attained, the truth or falsehood of which is already known, whence the correctness or absurdity of the supposition becomes established; or, in case of a problem, such relations are determined as prove the possibility or impossibility of the solution. In synthesis, observes Montucla, we proceed from the known to the unknown, from the trunk to the branches; in analysis, we proceed from the unknown to to the known, from the branches to the trunk.

The analytical method is frequently indispensable, when new problems are to be solved, or new theorems investigated. No doubt, many propositions which the ancients have transmitted to us in the synthetical form, owe their discovery to analysis.

Of this method of procedure, the mathematical collections of Pappus, and the work, *De Sectiones Rationis*, of Appollonius, furnish the principal specimens which the ancient geometers have left us.

In the ancient geometry, the magnitudes under consideration, were mostly presented to the mind through the medium of representations, as similar as practicable to their antitypes. In some instances, however, this analogy was entirely abandoned, as in the fifth book of Euclid's Elements, where right lines are the only repre-

sentatives used, yet the reasoning is so conducted, as to be equally applicable to magnitudes of every kind, and even to abstract numbers. In this instance, we may discover a commencement of that species of generalization, which forms so conspicuous a feature in our modern mathematics.

The ancient analysis furnished the germ of that branch of mathematics, which, in the hands of the moderns, has become the great master key to all the rest; but it does not appear to have assumed the character of a distinct science, till after the commencement of the Christian era.

The earliest writer, in whose works the science of algebra is distinctly seen, was Diophantus, a mathematician of the Alexandrian school. The time in which he lived is not precisely known, but it was not later than the fourth century, as the daughter of Theon, the amiable and accomplished Hypathia, who died about the beginning of the fifth, wrote a commentary on his works.

Whether Diophantus was the inventor, or only an improver of algebra, cannot now be known; the latter supposition, however, is the more probable, as the science, in his hands, exhibits a degree of maturity, which it can hardly be supposed to have attained in the first period of its existence.

A part only of the original work of Diophantus is now to be found; in this he does not explain the first principles of the science, but teaches the solution of a great variety of difficult questions, in that branch of the subject, now called, from him, the Diophantine Algebra, or the *indeterminate analysis*, applied to equations of the higher orders. The various questions, if original, which he has formed, and the address with which he has conducted their solutions, necessarily inspire his readers with a high opinion of his invention and discernment. His work, written in the original Greek, was discovered in the Vatican Library, about the middle of the sixteenth century.

Though the science of algebra appears to have originated among the Greeks of the Alexandrian school, the inhabitants of western Europe derived their knowledge

of it from the Arabs, who are by some supposed to have been its inventors.

Dr. Wallis observes, that they differ essentially from Diophantus in their manner of expressing the powers. The Greek analyst calls the 2d, 3d, 4th, 5th 6th, &c. powers, the square, cube, squared square, squared cube, cubed cube, &c.; each power being designated by the two inferior powers of which it is the product. But the Arabian algebraists denominate the 5th power the first sursolid, the 6th the squared cube; being the square of the cube, and not the product of a square and cube; the 7th the second sursolid, and so of the other powers. Hence, he infers, that the Greek and Arabian analyses were not derived from a common source.

With due regard for the opinion of this eminent scholar, it appears quite as rational to suppose, that the Arabian mathematicians may have borrowed from their tutors, the Greeks, this branch, with the rest of the mutilated sciences, and adopted, in the denomination of their powers, a mode of expression peculiar to themselves, as to believe, that while the other parts of Grecian science were sought with avidity, this was permitted to sleep amidst the dust of neglected libraries, till the same thing had been reinvented by a people much less advanced in scientific knowledge, and less remarkable for invention than their predecessors. In the history of scientific discoveries, there always appears a regular dependence in the successive stages. The discoveries of one age are the results of those made in the former. Logarithms were invented, when the discoveries in astronomy and trigonometry had rendered their use indispensable. The discoveries of Newton could not have been made, even by that gigantic genius, in the time of Copernicus; nor could Columbus have led his trembling companions across the Atlantic, before the invention of the mariners' compass.

Whether the science of algebra was *invented* by the Arabs, or borrowed from the Greeks, the name is unquestionably of Arabic origin. The names given by the Arabs, for they used a plurality, were *algebra v'almucabala.* These words, according to Lucas de Burgo, sig-

nify *restauratio et oppositio*, restoration or rebuilding, and opposition. Golius defines the word gebera, or giabera by *religavit consolidavit*, it bound or consolidated; and mocabulat, by *comparatio, oppositio*, comparison, opposition.

By these words they probably designed to indicate the general objects of the science. The quantity whose value is sought, is commonly interwoven with, or *bound* to other quantities, in such a manner as to form one or more equations, or *comparisons* of quantities set in *opposition* to each other. These equations are then transformed, or *rebuilt*, till the unknown quantity is brought out in *opposition* to a given or known one. The name *almucabala* was adopted by some of the Italian writers, and it is thus designated in some of the works of Cardan; but the term algebra appears now irrevocably fixed upon it.

The most ancient authors on algebra among the Arabs, are Mohammed ben Musa, and Thebit ben Corah. The former is described by Cardan as the inventor of the method of solving equations of the second degree, a discovery in which he was certainly anticipated by the mathematicians of the Alexandrian school. From the title given to his book, it has been inferred, that he flourished during the reign of Almamon, or in the early part of the ninth century.

Whether the Arabian algebraists proceeded beyond the solution of equations of the second degree, is an unsettled question. An accurate knowledge of the mathematical sciences, is seldom combined, in the same individual, with an extensive acquaintance with Arabic literature, and therefore, little is certainly known on this subject. The Bodleian Library in England, and that of the Escurial in Spain, are said to possess a great number of Arabic works on the subject of algebra.

Leonard of Pisa, who lived near the beginning of the thirteenth century, impelled by a thirst for mathematical knowledge, travelled into Arabia and other parts of the east, and on his return, first communicated to his countrymen the science of algebra. I do not find that

any of his writings on that subject have ever been published.

In a treatise upon trigonometry, by Regiomontanus, of Franconia, written about the year 1464, some problems are solved by algebra, in which he refers to the rules, as though commonly known.

But the earliest European author whose works, specially on this subject, have been published, was Lucas de Burgo, before mentioned. He was a franciscan, who travelled in the east, either in pursuit of knowledge, or for some purpose not now known, and after his return, taught mathematics at Naples, Venice and Milan. His work, in which the rules of algebra are laid down, was first printed in 1494. In this the science appears very far below our modern algebra. The rules for the solution of adfected quadratic equations, are given in semi-barbarous Latin verse, and the different cases separately treated. His solutions do not rise to equations above the second degree.

The solution of cubic equations appears to have been first effected by Scipo Ferrei, professor of mathematics at Bologna, about the beginning of the sixteenth century.

This solution, however, included but one case, namely, that in which the first and third powers only, of the unknown quantity, were involved. Some questions, including this case, being afterwards proposed to Nicholas Tartaglia, an eminent mathematician of Brescia, he discovered a general solution of equations of the third degree. This discovery he communicated to Jerome Cardan, a physician of Milan, under an injunction of secrecy. Cardan, notwithstanding, having found the demonstration, published it in his work, De Arte Magna, in 1545. This is now commonly called the method of Cardan.

Cardan first remarked the plurality of roots in quadratic equations, and their distinction into positive and negative. The solution of equations of the fourth degree was accomplished soon after the discovery of Tartaglia, by Lewis Ferrari, a pupil and coadjutor of Cardan.

Some intricacies belonging to equations of the third degree were further unravelled by Raphael Bombelli, of Bologna, and published in 1579.

The algebraists, whose labours have been noted, expressed known quantities by their proper numerical characters, and therefore, their solutions were destitute of that generality which constitutes so prominent a feature of our present analysis. Their *modes* of solution were applicable to all similar problems, but their *results* were confined to particular questions. Francis Vieta, who was born at Fontenoy in Poitou, in 1540, and died in 1603, by using letters of the alphabet to represent known as well as unknown quantities, gave an extent and generality to the science which it did not possess before. By this change, algebraists were enabled to include, in a single solution, a whole class of problems, and to obtain the result in each particular case, by simple substitution. He taught the method of taking, from an equation, its second term, and thus reducing adfected quadratics, to simple quadratics; and all cases of cubic equations to the case solved by Ferrei. He also taught the solution of cubic equations, having three possible roots, by the trisection of an angle. He made numerous improvements in algebra, and furnished the germs of some discoveries which have since grown up under other names.

Thomas Harriott, an English analyst, followed in the steps of Vieta, and made several important improvements in the science. He, first, adopted the plan of placing all the terms of an equation on one side of the sign of equality, and zero on the other; and showed that an equation thus expressed, may be always formed by the multiplication of binomial factors. This naturally led to the discovery, that every equation has as many roots, or values of the unknown quantity, as there are units in the index of its highest power. Harriott, notwithstanding this observation lay directly in his road, does not appear to have made it. The complete developement of negative and impossible roots, was left to exercise the ingenuity of succeeding inquirers. Vieta employed the large letters of the alphabet, and indicated

their powers by initials placed over them, as exponents are now used: Harriott substituted small letters, and denoted their powers by repetitions of the letter; thus instead of A^q, A^c, &c. he wrote *aa, aaa,* &c., a small change indeed, but still an obvious improvement in the notation.

His Artis Analyticæ Praxis, containing his discoveries, was published after his death.* He was born at Oxford in 1560, and died in 1621. The celebrated French philosopher, René Descartes, contributed largely to the advancement of algebra. He explained the nature of *negative* roots, and taught the manner of finding their number by the changes of the signs in the general equation. The use of exponents, as now applied, is attributed to him; as is also the method of indeterminate co-efficients.

The first application of algebra to geometry, was long prior to the time of Descartes, yet those sciences are indebted to him for that intimate union, which has since contributed so extensively to the improvement of both. Descartes was born in 1596, and died in 1650.

Soon after the time of Descartes, the analytical science took a flight, which if followed, would lead me far beyond the bounds of a preface. This sketch of the history will, therefore, be closed with the remark, that this science, as enriched by the discoveries of Newton, Leibnitz, and others, has become in the hands of our modern philosophers, the torch to guide them through the most intricate labyrinths of science; that by its light they have traced the motions of the celestial bodies through all their mystic dance, and penetrated many of the recesses of nature, where, without its aid, they must have been bewildered and lost.

The following work was undertaken from a persuasion, that the books on algebra, used in our schools, were none of them, entirely adapted to the wants of a large class of pupils, many of whom do not enjoy the leisure,

* Bossut says it was published in 1620, Montucla and others say in 1631.

or the talents requisite for penetrating the depths of science, and yet are desirous of attaining a knowledge of this subject, sufficient to qualify them for studying successfully the common practical branches of mathematics, to which this serves as a key. In the most popular treatise on this science, with which our schools have been furnished, the progress of the student appears to me, needlessly obstructed, by difficulties near the commencement, which to a common intellect, are almost insuperable.

My object has been to present the most useful parts of the science in such order, that no very abstruse process should be required, before the pupil had been sufficiently exercised, to acquire the requisite skill. The expedients demanded for solving the questions, are mostly pointed out before they are called for in practice.

In a popular treatise on a subject which has engaged the attention of so great a number of authors, many of them, unquestionably, endued with talents of the highest order, it would be idle to expect much originality. My object has been to smooth the path of the student, and diminish the toil of the tutor. The work, with all its imperfections, is submitted to the inspection of the public.

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THE

PRACTICAL ANALYST.

DEFINITIONS.

ARTICLE 1. Algebra, or specious arithmetic, is the science of computing by symbols or general characters.

2. Quantities, of whatever kind, are usually denoted by letters of the alphabet.

3. The relations of quantities, and the operations to be performed on them, are indicated by the following characters, thus:

4. The sign + *plus*, or *more*, indicates addition, as $a+b$, signifies that b is added to a.

5. The symbol — *minus*, or *less*, indicates subtraction; thus, $a-b$, signifies that b is subtracted from a.

The characters + and — are called, by way of eminence, the *signs* of the quantities to which they are prefixed.

6. Multiplication is denoted by the sign ×, *into*, placed between the factors, as $a \times b$; or by a period, as $a.b$; or, more frequently, by joining the letters, like letters in a word, as ab; each of which expressions denotes the product of a and b.

7. Division is indicated by the sign ÷ *by*, placed between the terms; or by writing the dividend above, and the divisor below a horizontal line; thus $a \div b$, or $\frac{a}{b}$ denotes the quotient of a when divided by b.

8. The difference of two quantities, when it is unknown which is the greater, is indicated by the sign $\backsim$, placed between them; thus, $a \backsim b$, or $b \backsim a$, denotes the difference of a and b.

9. : :: : indicate *proportion;* thus, $a : b :: c : d$, may be read, a has to b the same ratio that c has to d.

10. $=$ *equal to,* signifies that the quantities between which it is placed, are equal to each other; thus, $b+c=a-d+e$, is a combination, called an *equation,* which signifies, that when the operations indicated by the signs are performed on the quantities placed on each side of the sign of equality, the results are equal to each other.

11. A *simple quantity* is that which consists of one term only, viz. a quantity denoted by a single letter, or several letters and figures, connected by the sign of multiplication or division, expressed or understood,* as a, abc, $5cd$; $\frac{cb}{d}$.

12. A *compound quantity* consists of two or more simple quantities connected by the signs of addition or subtraction, as $a-b$, $ab-ac+bd$.

13. A *root* is a number or quantity, from which a power is conceived to arise.

14. A *power* of a number or root, is the product of a unit, multiplied continually by the given root, any proposed number of times; and the figure or quantity which indicates the number of multiplications thus made, is called the *index* or *exponent;* thus, $1\times5\times5\times5\times5=625$; and, $1\times a\times a\times a\times a\times a$, or $aaaaa$, are the 4th and 5th powers of 5 and a respectively, and are usually expressed by the root with the index of the power set over it; as 5^4, a^5; hence, $a^0=1$, whatever value may be assigned to a.

15. The second power is called the *square;* the third power, the *cube;* the fourth power, the *biquadrate,* &c. of their respective roots.

---

* An absolute number, though containing numerous digits, is considered as a simple quantity.

16. The *radical sign** $\surd$, prefixed to a quantity, indicates the square root; $\sqrt[3]{}$, the cube root; $\sqrt[4]{}$, the fourth root, &c. Roots are also expressed by fractional exponents; thus, $\surd a$ or $a^{\frac{1}{2}}$, denotes the square root of a; $\sqrt[3]{a}$, or $a^{\frac{1}{3}}$, the cube root of a; and $\sqrt[3]{a^2}$, or $a^{\frac{2}{3}}$, the cube root of the square of a.

17. A root which cannot be accurately expressed in numbers, is called a *surd*, or *irrational* quantity; as $\surd 5$, $\sqrt[3]{5}$.

18. A quantity which has no radical sign, or which having a radical sign, admits of an accurate extraction of the root indicated by the sign, is called *rational;* thus a, $\surd 16$, $\sqrt[3]{b^3}$, are rational quantities.

19. When a compound quantity has a line drawn over it, or is enclosed in brackets, the operation indicated by a preceding or subsequent sign, is to be performed on the whole considered as a simple quantity; thus, $\overline{a+b-c}\times d$, and $(b+c)\times(d-e)$, signify that the compound quantities connected by the sign $\times$, are multiplied together; and $\surd ab+dc$, $\sqrt[3]{\{ab-cd+ef,\}}$ signify the square root, and the cube root respectively of the compound quantities, which are preceded by the radical signs.

20. A number prefixed to a letter, or combination of letters, is called the *co-efficient:* thus, in the expression $3ab$, 3 is the co-efficient.

21. A compound quantity, consisting of two terms, is called a *binomial*, as $a+b$; one of three terms, a *trinomial*, as $ab+ac+de$.

22. *Like quantities* are those which consist of the same letters similarly involved; as ab, $3ab$, $5ab$.

23. *Unlike quantities* consist of different letters, or different powers of the same letters, as a, ab, $3a^2$, $4ab^3$.

In the solution of problems, it is usual to denote known or given quantities, by the initial letters, a, b, c, &c., and unknown ones, by the final letters, v, x, y, &c.

---

* So called from *radix*, a root.

The following examples are given for the purpose of exercising the student in the application of the algebraic signs.

Required the numerical values of the following combinations, supposing $a=7$, $b=6$, $c=5$, $d=3$, $e=2$.

$a^3+4bc-3de=343+120-18=445.$

$(ab+cd)\times(3bc+4ad)=(42+15)\times(90+84)=57\times174=9918.$

$a^2b+cd^2-3be=303.$

$b^3+a^2d-2ce-4bc=223,$

$(ac^3+2bd)\times5ac=159425.$

$\frac{b^3}{6a+de^2}=4$ $\qquad$ $\sqrt{a^3+b^2+ad}=20$

$\frac{a^2b+13cd^3}{2a^2-3de}=47.662538$

$\frac{5ad^3-2ace^2}{7b+3d}=13\frac{2}{51}.$

$\sqrt[3]{4a^3+7d^3+53be}=13$

Section I.

ADDITION.

Case 1.

24. *When the quantities are like, and have like signs.*

Add all the co-efficients together, and to their sum annex the given literal expression, prefixing the common sign.*

* The sign + or −, is frequently prefixed to quantities which are not preceded by others, to or from which, they are required to be added or subtracted: a preceding quantity, however, may always be supposed. Algebraic quantities are said to be positive or negative, according as they are preceded by the sign + or −; the former being always understood, where none is expressed.

EXAMPLES.

$3a$	$2be$	$2a^2b$	$-3xy^2$	$2a-3b$
$5a$	$3be$	$3a^2b$	$-xy^2$	$5a-6b$
$7a$	be	$4a^2b$	$-4xy^2$	$7a-5b$
$4a$	$5be$	a^2b	$-7xy^2$	$8a-4b$
$6a$	$8be$	$5a^2b$	$-9xy^2$	$9a-4b$
a	$9be$	a^2b	$-11xy^2$	$15a-b$
$26a$				

$5x^2y-7bc+ab$	$6xy\frac{1}{2}-7ac$	$4ab^3+3a^2c-bc^3$
$3x^2y-6bc+4ab$	$3xy\frac{1}{2}-3ac$	$5ab^3+4a^2c-bc^3$
$5x^2y-3bc+ab$	$4xy\frac{1}{2}-5ac$	$3ab^3+5a^2c-2bc^3$
$7x^2y-\frac{1}{2}bc+4ab$	$3xy\frac{1}{2}-2ac$	$8ab^3+8a^2c-8bc^3$
$8x^2y-2\frac{1}{2}bc+3ab$	$8xy\frac{1}{2}-ac$	$4ab^3+3a^2c-7bc^3$
$3x^2y-7bc+6ab$	$5xy\frac{1}{2}-7ac$	$ab^3+6a^2c-3bc^3$
$4x^2y-3bc+2ab$	$3xy\frac{1}{2}-2ac$	$5ab^3+5a^2c-4bc^3$
$5x^2y-4bc+3ab$	$4xy^2-13ac$	$2ab^3+a^2c-5bc^3$

CASE 2.

25. *When the quantities are like, but have unlike signs.*

Take the difference between the sum of the positive and the sum of the negative co-efficients, prefix the sign of the greater, and annex the literal part.

EXAMPLES.

$4ab^2-3cd^2$	x^2y^3	$5a^3b-7x^2y^2$
$2ab^2+6cd^2$	$-3x^2y^3$	$-3a^3b+4x^2y^2$
$-5ab^2+4cd^2$	$5x^2y^3$	$4a^3b+2x^2y^2$
ab^2-7cd^2	$7x^2y^3$	$-8a^3b+x^2y^2$
$-3ab^2+5cd^2$	$-4x^2y^3$	$7a^3b-3x^2y^2$
$6ab^2-cd^2$	$-9x^2y^3$	$-a^3b+5x^2y^2$
$5ab^2+4cd^2$	$-3x^2y^3$	

$3ab^3-5c^2d^2$	$8\surd ay$	$20ax-7b\surd xy$
$-5ab^3+2c^2d^2$	$-9\surd ay$	$-31ax+5b\surd xy$
$-ab^3+4c^2d^2$	$-7\surd ay$	$27ax-6b\surd xy$
$7ab^3-3c^2d^2$	$3\surd ay$	$18ax-5b\surd xy$
$8ab^3-5c^2d^2$	$5\surd ay$	$-4ax+13b\surd xy$
$-6ab^3+7c^2d^2$	$-4\surd ay$	$-30ax+11b\surd xy$

Case 3.

26. *When there are unlike quantities, and different signs.*

Collect the like quantities as in the foregoing cases, and connect the results by their proper signs.

EXAMPLES.

$4ax-130+3x^{\frac{1}{2}}$	$7ax$	$3a^2-7be+5ad$
$5x^2+3ax+9x^2$	$-5xy$	$2xy-7ad-5a^2$
$7xy-4x^{\frac{1}{2}}+90$	$-3ax$	$3ad-6xy-3bc$
$6x^2-5\surd x+3xy$	$2xy$	$5xy+2a^2-4ad$
$40-5ax-10xy$	$-3ab$	$4bc+3ad-xy$
$2ax+20x^2-6x^{\frac{1}{2}}$	$4ax-3xy-3ab$	

$5x^2y-7ax-3bc$	$14ax-2x^2$
$2bc+3x^2y+4ax$	$5ax+3xy$
$3ax-5bc+7ad^2$	$8y^2-4ax$
$2bc+3ad^2-6ax$	$3x^2+26$
$x^2y-5bc+4ax$	$4xy-5y^2$
$8ad^2-3ax-4x^2y$	$5x^2-8ax$
$ax-18ad^2-3bc$	$7y^2-26$

$x^3y^2+4ad^3$	$-7b^2c^2$	
$5b^2c^2-7ad^3$	$+2ac^4$	
$3ac^4$	$+6b^2c^2$	$+5x^3y^2$
$8ad^3$	$-5ac^4$	$-6b^2c^2$
$5x^3y^2+ad^3$	$-b^2c^2$	
$3b^2c^2$	$-5x^3y^2$	$-2ac^4$
$6x^3y^2+2ac^4$	$-6ad^3$	

1. Add $3a+b$, $7a-5b$, $7b-3a$, $4a-2b$, and $8a-5b$ together. Result, $19a-4b$.

2. Add $5ax+7bc-3a^2$, $8bc+4a^2-2ax$, $11a^2-4bc-6ax$ and $2ax-3bc+6a^2$ into one sum.

Result, $8bc+18a^2-ax$.

3. Add together the following quantities, $7abc+5a^2d-3xy^2$, $2a^2d+5xy^2-3abc$, $9bc^2+8xy^2+3a^2d$, $4bc^2-3abc-a^2d$, and $a^2d-10xy^2-abc-7bc^2$.

Result, $10a^2d+6bc^2$.

SUBTRACTION.

27. Change the signs of the quantities which are to be subtracted, or conceive them to be changed, then collect the terms as in addition.

EXAMPLES.

From $8xy-3+6x-y$	$3x^2-2ab$	$5ab^2-7b^2c+ad$
Take $3xy-7-6x+5y$	x^2-7ab	$3ab^2-2b^2c-4ad$
$5xy+4+12x-6y$		

From $17ab-3xy+5b^2$	$8ax-5bc$	$9aey-4bc$
Take $11ab+4xy+7b^2$	$7ax-7bc$	$2aey+7bc$

From $7x^2-2a\sqrt{x}-18+5bc$	$4ad-7xy+3bc$
Take $3x^2-7a\sqrt{x}-9-3bc$	$5ad-y^2z+5bc$

1. From $a+b$ take $a-b$. Result $2b$.

2. From $8a-12x$ take $5a+3x$. Result $3a-15x$.

3. From $12a+10b+13ax-3ab$, take $7a-5b+3ax$.
Result $5a+15b+10ax-3ab$.

4. From the sum of $3ab-7ax$ and $7ab+3ax$, take $4ab-3ax-4xy$. Result $6ab-ax+4xy$.

5. From $5x^2-4xy+5$, subtract $4x^2-4xy+9$.
Result x^2-4.

6. From ax^3-bx^2+x subtract px^3-cx^2+ex.
Result, $(a-p)x^3+(c-b)x^2+(1-e)x$.

7. From bx^3+cx^2-dx+e take px^3-qx^2+rx-s.
Result, $(b-p)x^3+(c+q)x^2-(d+r)x+e+s$.

MULTIPLICATION.

Case 1.

28. *When the factors are both simple quantities.*

Multiply the coefficients together, and annex all the letters to the product.*

Note.—When the signs of the factors are like, the product must be made positive; when unlike, negative.

EXAMPLES.

$10ab$	$2xy^3$	$37a^2b$	$5ad$
$5a^2c$	$-6x^2y$	$3ab^2c$	$3a^2b^3c$
$50a^3bc$	$-12x^3y^4$		

* From. Def. 14, it is obvious, that the product of two or more powers of any letter, may be expressed by that letter with an index equal to the indices of the factors; thus, $a^2a^2=aaa.aa=a^5$.

$-acx^2$	$3xyz$	$5a^3bc$	$-4ad^2$	$5x^3y^2$
$-bc^2x$	$-2x^2y$	$7ac^3$	$-5a^2b$	$-4x^2y^3$

Case 2.

29. *When one of the factors is a compound quantity.*

Multiply the simple factor into each term of the compound one, and connect the products by their proper signs.

EXAMPLES.

$a^2-2ab+b^2$	$xy+2y^2$	$2ab-2b^2$
a^2	$3ax$	$-4b^2$
$a^4-2a^3b+a^2b^2$		

xy^2+4bc^2-9	$8x^3y+5ay^3-7bc^2$	$3c^2+ax$
$5cx$	$-2a^2c$	$5ax$

Case 3.

30. *When both factors are compound quantities.*

Multiply the whole multiplicand by every term of the multiplier, and collect the several products as in addition.

EXAMPLES.

$a+b$	a^2+ab+b^2
$a+b$	a^2-ab+b^2
a^2+ab	$a^4+a^3b+a^2b^2$
$ab+b^2$	$-a^3b-a^2b^2-ab^3$
	$a^2b^2+ab^3+b^4$
$a^2+2ab+b^2$	$a^4+a^2b^2+b^4$

$a+b$ $\quad x^2+xy+y^2$ $\quad x^4+x^3y+x^2y^2+xy^3+y^4$
$a-b$ $\quad x-y$ $\quad x-y$

$2x^2+3xy-4y^2$ $\quad 2ab-3ac+4bc$ $\quad x^2-xy+y^2$
$3x-5y$ $\quad ab+ac-bc$ $\quad x-y$

1. Multiply $5a^2+4b^2$ by $5a^2-4b^2$.
Result, $25a^4-16b^4$

2. Multiply $x^3+3x^2y+3xy^2+y^3$ by $x^3+2xy+y^2$.
Result, $x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$.

3. Multiply $3x^2+2xy+5y^2$ by x^2-xy+y^2.
Result, $3x^4-x^3y+6x^2y^2-3xy^3+5y^4$.

4. Multiply $5x^2+10xy+5y^2$ by $5x^2-10xy+5y^2$.
Result, $25x^4-50x^2y^2+25y^4$.

5. Multiply $a^3+3a^2b+3ab^2+b^3$ by $a^3-3a^2b+3ab^2-b^3$.
Result, $a^6-3a^4b^2+3a^2b^4-b^6$.

6. Multiply a^2-ab+b^2-bc by a^2+ab-b^2+bc.
Result, $a^4-a^2b^2+2ab^3+2b^3c-b^2c^2-b^4$.

7. Multiply $x^4-x^3y+x^2y^2-xy^3+y^4$ by x^2+xy+y^2.
Result, $x^6+x^4y^2-x^3y^3+x^2y^4+y^6$.

8. Multiply $x+y$ by $x-y$, and the product by x^2+y^2.
Result, x^4-y^4.

9. Find the continued product of $a^2-2ab+b^2$, $a^2+2ab+b^2$, $a^2+a^2b+ab^2+b^3$ and $a-b$.
Result, $a^8-2a^6b^2+2a^2b^6-b^8$.

10. Find the continued product of $3x+6$, $3x+2$, $3x-2$ and $3x-6$. Result, $81x^4-360x^2+144$.

DIVISION.

Case 1.*

31. *When the divisor and dividend are both simple quantities.*

Divide the co-efficient of the dividend by that of the divisor; expunge from the dividend such letters as are common to it and the divisor when they have the same exponents; when the exponents are not the same, subtract the exponent of the divisor from that of the dividend, and use the remainder as the index of the letter in the quotient; write in the quotient the letters which have not been expunged, with the co-efficient above determined.

Note.—In division, as in multiplication, like signs in the divisor and dividend require the positive sign in the quotient; unlike signs, the negative.

EXAMPLES.

$3ab)9a^2bcd$ $\quad -5x^2y^3)15x^4y^3z^2$ $\quad 2ad)6a^3d^2c$

$3acd$ $\quad -3x^2z^2$

$17ab^2)51a^2b^5$ $\quad 3a^2c)12a^4be^3$ $\quad 7a^2b)-21a^4bc^2$

$-3axz)-18a^2xy^2z^2$ $\quad 15abc^3)-45a^2b^3c^4$

* If the student, who is just entering into this science, should express the powers by repetitions of the letters, the process will be more simple, and the grounds of the method indicated by the rule, be rendered obvious.

32. *When the dividend is not a multiple of the divisor.*

Set them down as a vulgar fraction, the divisor being the denominator; and divide the terms by such numbers and quantities as are common to both; the result will be the fractional answer.

$$\frac{15a^2bc}{10ab^2}=\frac{3ac}{2b} \qquad \frac{18axy^2}{-8ax^2y}=\frac{-9y}{4x^2}$$

Divide $28xy^3$ by $7x^2y^2$. Result, $\frac{4y}{x}$

Divide $-10abc$ by $15abd$ Result, $\frac{-2c}{3d}$

Divide $11mn$ by $22m^2$ Result, $\frac{n}{2m^2}$

Divide $19xyz$ by $25x^2yz^2$ Result, $\frac{19}{25xz}$

CASE 2.

When the dividend is a compound quantity, and the divisor a simple one.

Divide every term of the dividend by the divisor, as in the former case, and connect the results by their proper signs.

EXAMPLES.

$$\frac{18ax^2-15a^2y+27a^3}{3a}=6x^2-5ay+9a^2$$

1. Divide $15axy^2-30bx^2y+10x^2y^2$ by $5xy$.
3. Divide $30a^2c-15ac^2+18ac$ by $3ac$.
4. Divide $15a^2bc-15acx^2+5ad^2$ by $-5ac$.
5. Divide $21a^3x^3-7a^2x^2-14ax$ by $7ax$.
6. Divide $72a^2b^2y^2-12a^2by+36ab^3y$ by $6aby$.
7. Divide $28x^3y^2+35x^2y^4-7x^3y^4z$ by $7x^2y^2$.

Case 3.

33. *When the divisor and dividend are compound quantities.*

Arrange the terms, both of the divisor and dividend, in such manner, that the higher powers of some one letter shall always precede the lower; then find the first term of the quotient, by using the first of the divisor and dividend, as in the first case. Multiply the whole divisor by the quotient thus found; subtract the product from the dividend, as in common arithmetic; and proceed till the dividend is exhausted. The remainder, if any thing remain, with the divisor for a denominator, must be annexed, with its proper sign, to the quotient.

EXAMPLES.

$$
\begin{array}{l}
a+b)a^3+3a^2b+3ab^2+b^3(a^2+2ab+b^2 \\
\quad a^3+\ a^2b \\
\quad \hline
\quad\quad 2a^2b+3ab^2 \\
\quad\quad 2a^2b+2ab^2 \\
\quad\quad \hline
\quad\quad\quad\quad ab^2+b^3 \\
\quad\quad\quad\quad ab^2+b^3
\end{array}
$$

$$
\begin{array}{l}
a^2-2ab+b^2)a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-3b^5(a^3- \\
\quad a^5-2a^4b+\ a^3b^2 \qquad\qquad [3a^2b+3ab^2-b^3- \\
\quad \hline
\quad\quad -3a^4b+\ 9a^3b^2-10a^2b^3 \qquad\qquad 2b^5 \\
\quad\quad -3a^4b+\ 6a^3b^2-\ 3a^2b^3 \qquad\qquad a^2-2ab+b^2 \\
\quad\quad \hline
\quad\quad\quad\quad 3a^3b^2-\ 7a^2b^3+5ab^4 \\
\quad\quad\quad\quad 3a^3b^2-\ 6a^2b^3+3ab^4 \\
\quad\quad\quad\quad \hline
\quad\quad\quad\quad\quad\quad -a^2b^3+2ab^4-3b^5 \\
\quad\quad\quad\quad\quad\quad -a^2b^3+2ab^4-\ b^5 \\
\quad\quad\quad\quad\quad\quad \hline
\quad\quad\quad\quad\quad\quad\quad\quad -2b^5
\end{array}
$$

Divide $x^4-4x^3y+6x^2y^2-4xy^3+y^4$ by $x-y$.

Result, $x^3-3x^2y+3xy^2-y^3$.

Divide a^2-b^2 by $a-b$. Result, $a+b$.

Divide a^3+b^3 by $a+b$. Result, a^2-ab+b^2.

Divide $y^3-3y^2-4y+12$ by y^2-4. Result, $y-3$.

Divide $b^6-3b^4x^2+3b^2x^4-x^6$ by $b^3-3b^2x+3bx^2-x^3$.

Result, $b^3+3b^2x+3bx^2+x^3$.

Divide $24a^4-b^4$ by $3a-6b$.

Result, $8a^3+16a^2b+32ab^2+64b^3+\frac{383b^3}{3a-6b}$

Divide $a^4+4a^2b^2-32b^4$ by $a+2b$.

Result, $a^3-2a^2b+8ab^2-16b^3$.

Divide x^6+y^6 by $x+y$.

Result, $x^5-x^4y+x^3y^2-x^2y^3+xy^4-y^5+\frac{2y^3}{x+y}$

PROMISCUOUS EXAMPLES.

1. Required the sum of $5ac+7be$, $3ac-4be$, $7ac-3be$, $8ac+4be$, and $2ac-3be$. Ans. $25ac+be$.

2. Collect $3ax+5xy-7z^2$, $3xy-2ax+4z^2$, $7ax+3z^2-xy$, and $5z^2-4ax+2xy$, into one sum; and subtract therefrom the sum of the following quantities, $3ax-5xy+4z^2$, $7xy-5ax-6z^2$, and $5ax-2xy+6z^2$.

Result, $ax+9xy+z^2$.

3. Required the product of $x^3+x^2y+xy^2+y^3$ by $x-y$.

Ans. x^4-y^4.

4. What is the sum of the products of $x^3+3x^2y+3xy^2+y^3$ by $x+y$, and $x^2-2xy+y^2$ by $x^2-2xy+y^2$?

Ans. $2x^4+12x^2y^2+2y^4$.

5. Divide x^5-y^5 by $x-y$, and from the quotient subtract the product of x^2+y^2 by x^2-xy.

Result, $2x^3y+2xy^3+y^4$.

6. Multiply $3a^3-9a^2b+9ab^2-3b^3$ by $a^2+2ab+b^2$, and $a^3+3a^2b+3ab^2+b^3$ by $3a^2-6ab+3b^2$, and find the sum of the products. Result, $6a^5-12a^3b^2+6ab^4$.

7. If a^4-b^4, be divided by $a-b$, and the product of a^2+ab+b^2 by $a-b$, subtracted from the quotient, what will the remainder be? Ans. $a^2b+ab^2+2b^3$.

8. Divide $x^8+8x^7y+28x^6y^2+56x^5y^3+70x^4y^4+56x^3y^5+28x^2y^6+8xy^7+y^8$ by $x^3+2xy+y^2$; and multiply $x^4-4x^3y+6x^2y^2-4xy^3+y^4$ by $x^2-2xy+y^2$, and find the sum and difference of the results.

Result, { Sum, $2x^6+30x^4y^2+30x^2y^4+2y^6$.
Difference, $12x^5y+40x^3y^3+12xy^5$.

Section II.

INVOLUTION, OR THE RAISING OF POWERS.

34. *To involve a given quantity to any power.**

Multiply the quantity by itself as many times as there are units in the index of the given power diminished by one.

EXAMPLES.

Required the 5th power of a^3b^2c.

$$(a^3b^2c)^5=a^{15}b^{10}c^5$$

Required the 4th power of $2b-c$.

* When the quantity to be involved is a simple one, multiply its index, or indices by that of the power proposed, observing that the even powers of negative quantities are positive, and the odd ones negative.

$$
\begin{array}{l}
2b - c \\
2b - c \\
\hline
4b^2 - 2bc \\
\quad -2bc + c^2 \\
\hline
4b^2 - 4bc + c^2 \\
2b - c \\
\hline
8b^3 - 8b^2c + 2bc^2 \\
\quad - 4b^2c + 4bc^2 - c^3 \\
\hline
8b^3 - 12b^2c + 6bc^2 - c^3 \\
2b - c \\
\hline
16b^4 - 24b^3c + 12b^2c^2 - 2bc^3 \\
\quad - 8b^3c + 12b^2c^2 - 6bc^3 + c^4 \\
\hline
16b^4 - 32b^3c + 24b^2c^2 - 8bc^3 + c^4 \\
\hline
\end{array}
$$

Find the 3d power of $5a^2$.

Required the 5th power of $-2a$.

Required the 4th power of $a+x$.

Required the square of $3a-2b$.

Required the 6th power of $5a+b$.

Required the 3d power of $a+2b+c$.

35. A binomial is raised to any power, with great facility, by the following method.

Set down, for the first term of the power, the first term of the root involved to the given power.

The succeeding terms, without the co-efficients, consist of the successive powers of the first term of the root, regularly descending, joined to the powers of the second, regularly ascending from the first; the common difference of the indices being one.

The co-efficient of the second term is the index of the given power; and if a co-efficient, already found, be multiplied by the exponent of the leading quantity, or first

term of the root, contained in the same term of the power, and divided by the number of terms to that place; the quotient will be the co-efficient belonging to the next term.*

Note.—To determine the signs of the different terms, it must be remembered that the even powers of negative quantities are positive, and the odd powers negative.

EXAMPLES.

Required the 5th power of $a+x$.

$$a^5 \quad a^4x \quad a^3x^2 \quad a^2x^3 \quad ax^4 \quad x^5$$

The co-efficients 1, 5, $\frac{5\times4}{2}=10$, $\frac{10\times3}{3}=10$, &c.

Hence, $\overline{a+x}^5=a^5+5a^4x+10a^3x^2+10a^2x^3+5ax^4+x^5$.

Required the 6th power of $2x-3y$.

$$(2x)^6 \quad (2x)^5.3y \quad (2x)^4.(3y)^2 \quad (2x)^3.(3y)^3 \quad (2x)^2.(3y)^4$$
$$2x.(3y)^5 \quad (3y)^6$$

Or, $64x^6 \quad 96x^5y \quad 144x^4y^2 \quad 216x^3y^3 \quad 324x^2y^4 \quad 486xy^5 \quad 729y^6$

The co-efficients 1, 6, $\frac{6\times5}{2}=15$, $\frac{15\times4}{3}=20$, 15, 6, 1.

Hence, $(2x-3y)^6=64x^6-576x^5y+2160x^4y^2-4320x^3y^3+4860x^2y^4-2916xy^5+729y^6$.

2. Required the 3d power of $a+y$.

Ans. $a^3+3a^2y+3ay^2+y^3$.

3. Required the 4th power of $a-b$.

Ans. $a^4-4a^3b+6a^2b^2-4ab^3+b^4$.

* The co-efficients of any two terms, equally distant, the one from the beginning, and the other from the end, are alike; hence, the computation of the first half of them will supply the whole.

4. Required the 7th power of $x+y$.

Result, $x^7+7x^6y+21x^5y^2+35x^4y^3+$, &c.

5. Required the 8th power of $a-c$.

Result, $a^8-8a^7c+28a^6c^2-56a^5c^3+70a^4c^3$, &c.

6. Required the 4th power of $2+x$.

Result, $16+32x+24x^2+8x^3+x^4$.

7. Required the 3d power of $a-3b$.

Result, $a^3-9a^2b+27ab^2-27b^3$.

This method of involution is easily extended to trinomials, quadrinomials, &c. by considering two or more of the terms as a single compound one.

8. Required the 4th power of $a+b-c$.

Here considering $b-c$ as the second term of the binomial, we have $(a+b-c)^4=a^4+4a^3(b-c)+6a^2(b-c)^2+4a(b-c)^3+(b-c)^4$; and involving $b-c$, to the powers indicated, by the same method, we have $(b-c)^2=b^2-2bc+c^2$; $(b-c)^3=b^3-3b^2c+3bc^2-c^3$; $(b-c)^4=b^4-4b^3c+6b^2c^2-4bc^3+c^4$; consequently, $(a+b-c)^4=a^4+4a^3b-4a^3c+6a^2b^2-12a^2bc+6a^2c^2+4ab^3-12ab^2c+12abc^2-4ac^3+b^4-4b^3c+6b^2c^2-4bc^3+c^4$.

9. Required the 3d power of $x+y+z$.

Result, $x^3+3x^2y+3xy^2+y^3+3x^2z+6xyz+3y^2z+3xz^2+3yz^2+z^3$.

10. Required the 2d power of $a+b-c-d$.

Result, $a^2+2ab+b^2-2ac-2bc+c^2-2ad-2bd+2cd+d^2$.

11. Required the 4th power of $x+y-3z$.

Result, $x^4+4x^3y+6x^2y^2+4xy^3+y^4-12x^3z-36x^2yz-36xy^2z-12y^3z+54x^2z^2+108xyz^2+54y^2z^2-108xz^3-108yz^3+81z^4$.

12. Required the 5th power of $a+b+c$.

Result, $a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$
$+5a^4c+20a^3bc+30a^2b^2c+20ab^3c+5b^4c+$
$10a^3c^2+30a^2bc^2+30ab^2c^2+10b^3c^2+10a^2c^3$
$+20abc^3+10b^2c^3+5ac^4+5bc^4+c^5$.

EVOLUTION, OR THE EXTRACTION OF ROOTS.

Case 1.

36. *To extract the root of a simple quantity.*

Extract the root of the co-efficient, for the numeral part; and divide the index or indices of the literal part, for the exponents of the root.*

When the root cannot be extracted, it must be indicated, as in definition 16.

EXAMPLES.

1. Required the 4th root of $256a^4x^8$.

$\sqrt[4]{256}=\pm 4$, the co-efficient; $a^{\frac{4}{4}}x^{\frac{8}{4}}=ax^2$, the literal part.

Hence, the root required is $4ax^2$, or $-4ax^2$.

2. What is the 5th root of $-5a^{15}b^{10}$?

The 5th root of 5 is a surd, and must be indicated thus: $\sqrt[5]{5}$ or $5^{\frac{1}{5}}$.

Hence, the root required is $-5^{\frac{1}{5}}a^3b^2$ or $-a^3b^2\sqrt[5]{5}$.

* The odd roots have the same signs as their powers; but the even roots of positive quantities may be either positive or negative. The even roots of negative quantities are impossible. These impossible or imaginary roots, frequently become the subject of important investigations, as will appear in the sequel of this work.

3. Required the square root of $16a^2b^4$.

Result, $\pm 4ab^2$.

4. What is the square root of $625x^4y^8$?

Ans. $\pm 25x^2y^4$.

5. Required the cube root of $-125a^6b^3$.

Result, $-5a^2b$.

6. What is the 5th root of $-32a^{10}b^5c^{15}$?

Ans. $-2a^2bc^3$.

7. What is the 3d root of $7x^3y^3$? Ans. $x\sqrt[3]{7y^3}$.

8. What is the 4th root of $81a^4b^6$?

Ans. $\pm 3a\sqrt{b^3}$, or $\pm 3ab\sqrt{b}$.

9. Required the 5th root of $-243a^5b^6$.

Result, $-3ab\sqrt[5]{b}$.

Case 2.

37. *To extract the square root of a compound quantity.*

Arrange the terms according to the dimensions of some letter, beginning with the highest.

Take the square root of the first term for the first term of the root, and subtract its square from the given quantity. Double the root thus found, for a defective divisor, divide the first term of the above remainder by this defective divisor, and annex the result both to the root and to the divisor.

Multiply the divisor thus completed by the term of the root last obtained, and subtract the product from the former remainder.

Divide the first term of the remainder as before, for the next term of the root; add this last and the preceding term of the root, to the last complete divisor for a new divisor. Multiply, subtract, and proceed as before.

Note.—In these additions, regard must always be paid to the signs of the quantities.

EXAMPLES.

$$\begin{array}{l|l} 4a^2+12ab+9b^2-20ac-30bc+25c^2 & 2a+3b-5c \\ 4a^2 & \\ \hline 4a+3b & 12ab+9b^2 \\ \quad 3b-5c & 12ab+9b^2 \\ \hline 4a+6b+5c & -20ac-30bc+25c^2 \\ & -20ac-30bc+25c^2 \end{array}$$

2. Required the square root of $x^2+2xy+y^2$.
Result, $x+y$.

3. Required the square root of $a^4+4a^3+6a^2+4a+1$.
Result, a^2+2a+1.

4. Required the square root of $x^4-4ax^3+6a^2x^2-4a^3x+a^4$.
Result, $a^2-2ax+x^2$.

5. What is the square root of $16x^4+24x^3+89x^2+60x+100$?
Ans. $4x^2+3x+10$.

6. What is the square root of $4a^4-12a^3x+5a^2x^2+6ax^3+x^4$?
Ans. $2a^2-3ax-x^2$.

7. What is the 4th root of $a^4+12a^3b+54a^2b^2+108ab^3+81b^4$?*
Ans. $a+3b$.

8. What is the 4th root of $a^8-4a^6b^2+6a^4b^4-4a^2b^6+b^8$?
Ans. a^2-b^2.

CASE 3.

38. *To extract any root of a compound quantity.*

Arrange the terms as before directed; take the root of the first term for the first term of the root, and subtract its power from the given quantity.

Take for a divisor, twice this root, three times its square, four times its third power, five times its fourth power, &c., according as the 2d, 3d, 4th, 5th, &c. roots are required to be extracted; divide the first term of the remainder, and annex the result to the root before found.

* The 4th, 8th, 16th, &c. roots, may be obtained by 2, 3, 4, &c. extractions of the square root.

Involve the whole root, thus obtained, to the given power, and subtract from the given quantity.

Divide the first term of the last remainder by the former divisor, annex the result to the root, and proceed as before.

EXAMPLES.

1. Required the 3d root of $8a^3+36a^2b+54ab^2+27b^3-48a^2c-144abc-108b^2c+96ac^2+144bc^2-64c^3$.

$8a^3+36a^2b+54ab^2+27b^3-48a^2c-144abc-108b^2c+96ac^2+144bc^2-64c^3|(2a+3b-4c$

$8a^3$

$12a^2)\ 36a^2b$

$8a^3+36a^2b+54ab^2+27b^3=(2a+3b)^3$

$12a^2)\ -48a^2c$

$8a^3+36a^2b+54ab^2+27b^3-48a^2c-144abc-108b^2c+96ac^2+144bc^2-64c^3$

2. Required the 3d root of $27x^3-54x^2y+36xy^2-8y^3$.
Ans. $3x-2y$.

3. What is the 3d root of $x^6-6x^5+15x^4-20x^3+15x^2+6x+1$?
Ans. x^2-2x+1.

4. Required the 5th root of $32x^5-80x^4+80x^3-40x^2+10x-1$.
Result, $2x-1$.

5. Required the 4th root of $16a^4-96a^3x+216a^2x^2-216ax^3+81x^4$
Ans. $2a-3x$.

6. Find the 5th root of $x^{10}+10x^9+35x^8+40x^7-30x^6-68x^5+30x^4+40x^3-35x^2+10x-1$.
Result, x^2+2x-1.

39. *Note.*—The roots of compound quantities may sometimes be obtained, with great facility, by a method of trial. To effect which, it may be observed, that the power of a binomial consists of as many terms as there are units in the exponent, increased by one. Hence, if the number of terms of a quantity, whose root is to be extracted, exceeds this sum, we may conclude, that the root consists of more terms than two.

When the root is judged to be a binomial, take the roots of the extreme terms, and connect the results by the sign + or —; but when the root appears to contain more terms than two, take also the roots of one or more of the other given terms, and connect the roots thus obtained, by the signs + or —, as may be judged proper.

Involve this supposed root to the given power, and if the result agrees with the quantity given, the root is manifestly correct.

This expedient, however, is not likely to abridge the labour of a student in the early periods of his course.

Section III.

ALGEBRAIC FRACTIONS.

40. Algebraic fractions depend upon the same principles as those in common arithmetic, and are managed by

similar rules, proper regard being paid to the signs and algebraic modes of operation.

EXAMPLES.

1. Required the greatest common divisor of x^2-b^2x, and $x^2+2bx+b^2$.*

$$
\begin{array}{l}
x^2-b^2x)x^2+2bx+\ b^2(1\\
\quad x^2-b^2\quad x^2-\ b^2\\
\hline
\qquad 2bx+2b^2\\
\quad x+b)x^2-b^2(x-b\\
\qquad\quad x^2+bx\\
\qquad\quad\ \ \hline
\qquad\quad\ -bx-b^2\\
\qquad\quad\ -bx-b^2
\end{array}
$$

Hence, $x+b$ is the divisor required.

2. Required to reduce $\dfrac{3a^2-2a-1}{4a^3-2a^2-3a+1}$ to its lowest terms.

First to find a common divisor.

$$
\begin{array}{l}
3a^2-2a-1)\ 4a^3-2a^2-\ 3a+\ 1\\
\qquad\quad 3\\
\hline
12a^3-6a^2-\ 9a+\ 3(4a\\
12a^3-8a^2-\ 4a\\
\hline
\qquad\quad 2a^2-\ 5a+\ 3\\
\qquad\quad 3\\
\hline
\qquad\quad 6a^2-15a+\ 9(2\\
\qquad\quad 6a^2-\ 4a-\ 2\\
\hline
\qquad\quad\ -11a+11
\end{array}
$$

* In examples of this kind, we are frequently obliged to adopt expedients which are not usual in common arithmetic. When every term of the divisor has a factor, which does not contain a

$$\begin{array}{r} -11a+11 \;\big)\; 3a^2-2a-1 \;(3a+1 \\ \text{Com. div. } a-1 \;\big)\; 3a^2-3a \\ \hline a-1 \\ a-1 \\ \hline \end{array}$$

$$a+1 \;\Big|\; \frac{3a^2-2a-1}{4a^3-2a^2-3a+1} = \frac{3a+1}{4a^2+2a-1} \text{ Result.}$$

3. Reduce $\frac{a^5-a^3b^2}{a^4-b^4}$ to its lowest terms. Result, $\frac{a^3}{a^2+b^2}$

4. Reduce $\frac{x^2-1}{xy+y}$ to its lowest terms. Result, $\frac{x+1}{y}$

5. Reduce $\frac{5a^5+10a^4b+5a^3b^2}{a^3b+2a^2b^2+2ab^3+b^4}$ to its lowest terms.

Result, $\frac{5a^4+5a^3b}{a^2b+ab^2+b^3}$

6. Reduce $\frac{a^4-x^4}{a^3-a^2x-ax^2+x^3}$ to its lowest terms.

Result, $\frac{a^2+x^2}{a-x}$

7. Reduce $\frac{6a^2+7ax-3x^2}{6a^2+11ax+3x^2}$ to its lowest terms.

Result, $\frac{3a-x}{3a+x}$

8. Reduce $\frac{15x^3-x^2-x+3}{9x^4+2x^2+1}$ to its lowest terms.

Result, $\frac{5x+3}{3x^2+2x+1}$

---

letter or number common to every term of the dividend, we divide by it, and use the resulting quotient as a divisor. When the divisor has been thus reduced, it often happens, that the division is still impracticable; in which case, we multiply the dividend by such a number as will make its first term a multiple of the first of the divisor.

9. Reduce $\frac{a}{2b}, \frac{2c}{3d}, \frac{3x}{4a}$, to other equivalent fractions, having a common denominator.*

Result, $\frac{6a^2d}{12abd}, \frac{8abc}{12abd}, \frac{9bdx}{12abd}$.

10. Reduce $\frac{3b}{4a^2}, \frac{2c}{3a}$, and $\frac{d}{2a}$ to a common denominator.

Result, $\frac{9b}{12a^2}, \frac{8ac}{12a^2}, \frac{6ad}{12a^2}$

11. Reduce $\frac{1}{3}, \frac{3a^2}{4}$, and $\frac{2a^2+b^2}{a+b}$, to a common denominator.

Result, $\frac{4a+4b}{12a+12b}, \frac{9a^3+9a^2b}{12a+12b}, \frac{24a^2+12b^2}{12a+12b}$

12. Reduce $\frac{31}{40}, \frac{2x}{3}, \frac{a^2+2x}{a}$, to a common denominator.

Result, $\frac{93a}{120a}, \frac{80ax}{120a}, \frac{120a^2+240x}{120a}$

13. Reduce $a-\frac{2x}{3a}$, to an improper fraction.

Result, $\frac{3a^2-2x}{3a}$

14. Reduce $10+\frac{2x-8}{3x}$ to an improper fraction.

Result, $\frac{32x-8}{3x}$

15. Required an improper fraction equivalent to $1+2x-\frac{x-5}{7x}$.

Result, $\frac{14x^2+6x+5}{7x}$

---

* The *least* common denominator should always be found in examples of this kind.—See *Case 2, Red. of Vulgar Fractions, in my Treatise on Arithmetic.*

16. What is the equivalent improper fraction to $a-\frac{x^2-a^2}{a}$? Ans. $\frac{2a^2-x^2}{a}$

17. Reduce $\frac{3ax+4x^2}{a+x}$ to a whole or mixed quantity.

Result, $3x+\frac{x^2}{a+x}$

18. Reduce $\frac{2a^2+2b^2}{a-b}$ to a whole or mixed quantity.

Result, $2a+2b+\frac{4b^2}{a-b}$

19. Reduce $\frac{27a^2-3b^2-4x+9a^2}{9a^2}$ to a mixed quantity.

Result, $3a+1-\frac{3b^2+4x}{9a^2}$

20. Reduce $\frac{x^3+3ax^2-a^3-b}{x^3+a^3}$ to a whole or mixed quantity. Result, $x^3-a^3+\frac{3ax^2-b}{x^3+a^3}$

21. Add $\frac{a}{2b}$, $\frac{2a}{3b}$, and $\frac{5b}{6a}$ together.

Sum, $\frac{6a^2}{12ab}+\frac{8a^2}{12ab}+\frac{10b^2}{12ab}=\frac{14a^2+10b^2}{12ab}=\frac{7a^2+5b^2}{6ab}$

22. Required the sum of $a-\frac{3x^2}{b}$ and $b+\frac{2ax}{c}$.

Result, $a+b+\frac{2abx-3cx^2}{bc}$

23. What is the sum of $2a+\frac{a+3}{5}$, and $4a+\frac{2a-5}{4}$?

Ans. $6a+\frac{14a-13}{20}$

24. Add $\frac{a+b}{a-b}$ to $\frac{a-b}{a+b}$. Sum $\frac{2a^2+2b^2}{a^2-b^2}$ or $2+\frac{4b^2}{a^2-b^2}$

25. What is the sum of $2a$, $\frac{a}{a-x}$ and $\frac{a-x}{a+x}$?

Ans. $2a+2+\frac{3x^2-ax}{a^2-x^2}$.

26. From $\frac{3a}{4}$ subtract $\frac{4a}{7}$ Remainder, $\frac{5a}{28}$

27. From $\frac{3ax}{4}$ take $\frac{2ax}{8}$. Rem. $\frac{ax}{12}$

28. From $\frac{2a-b}{4c}$ subtract $\frac{3a-4b}{3b}$.

Rem. $\frac{6ab-3b^2-12ac+16bc}{12bc}$

29. From $4a+\frac{2a-b}{3c}$ subtract $2a-\frac{4b-3a}{5a+c}$.

Rem. $2a+\frac{10a^2-5ab-7ac+11bc}{15ac+3c^2}$

30. Subtract $\frac{a+x}{a(a-x)}$ from $a+\frac{a-x}{a(a+x)}$. Rem. $a+\frac{4x}{a^2-x^2}$

31. Required the continued product of $\frac{a}{x}$, $\frac{2x}{9a}$, and $\frac{a-x}{a+x}$. Product, $\frac{2a-2x}{9a+9x}$

32. Required the product of $\frac{a+1}{ab+c^2}$, and $\frac{ac-c}{a^2-1}$.

Prod. $\frac{c}{ab+c^2}$

33. Required the continued product of $a-\frac{b^2}{a}$, $\frac{a}{a+b}$, $\frac{3x}{a-b}$, and $\frac{7b}{bx-x^2}$. Prod. $\frac{21b}{b-x}$

34. Required the continued product of $\frac{a^2-x^2}{a+b}$, $\frac{a^2-b^2}{ax+x^2}$, and $x+\frac{ax}{a-x}$. Prod. $2a^2-ax-2ab+bx$.

35. Divide $\frac{34a}{65b}$ by $\frac{17ax}{26bc}$. Quotient, $\frac{4c}{5x}$

36. Divide $\frac{26x^2}{35ay}$ by $\frac{13x}{25yz}$. Quot. $\frac{10xz}{7a}$

37. Divide $a-\frac{a^2-ax}{a+x}$ by $a+\frac{a^2+ax}{a-x}$. Quot. $\frac{ax-x^2}{a^2+ax}$

38. Divide $\frac{a-x}{a+x}$ by $\frac{a^2-x^2}{a^2+2ax+x^2}$. Quot. 1.

39. Divide $\frac{3x+1}{9}$ by $\frac{12x+4}{27x}$. Quot. $\frac{3x}{4}$

40. Divide $\frac{2a-b}{12cd}$ by $\frac{5ac}{6d}$. Quot. $\frac{2a-b}{10ac^2}$

41. Divide $\frac{5a^4-5b^4}{2a^2-4ab+2b^2}$ by $\frac{5a^2+5ab}{4a-4b}$.

Quot. $2a+\frac{2b^2}{a}$

42. Divide $\frac{x^4-b^4}{x^2-2bx+b^2}$ by $\frac{x^2+bx}{x-b}$. Quot. $x+\frac{b^2}{x}$

43. Divide $\frac{a^3-x^3}{a+x}$ by $\frac{a-x}{a^2+2ax+x^2}$.

Quot. $a^3+2a^2x+2ax^2+x^3$.

44. Divide $x^4-\frac{13}{6}x^3+x^2+\frac{4}{3}x-2$ by $\frac{4}{3}x-2$.

Quot. $\frac{3}{4}x^3-\frac{1}{2}x^2+1$.

45. Required the 4th power of $\frac{-2x}{3a}$. Result, $\frac{16x^4}{81a^4}$

46. Required the 3d power of $\frac{a-b}{a+b}$.

Result, $1-\frac{6a^2b+2b^3}{a^3+3a^2b+3ab^2+b^3}$

47. What is the 4th power of $1-\frac{2x}{a}$?

Ans. $1-\frac{8x}{a}+\frac{24x^2}{a^2}-\frac{32x^3}{a^3}+\frac{16x^4}{a^4}$

48. What is the cube root of $\frac{64x^3}{27a^6b^3}$? Ans. $\frac{4x}{3a^2b}$

49. Required the square root of $\frac{9a^6}{4x^2y^4}$.

Result, $\pm\frac{3a^3}{2xy^2}$

50. Required the square root of $x^4-3x^3+\frac{11}{4}x^2-\frac{3}{4}x+\frac{1}{16}$.

Result, $x^2-\frac{3}{2}x+\frac{1}{4}$

51. What is the 3d root of $\frac{1}{8}x^6-\frac{1}{2}x^5+\frac{41}{48}x^4-\frac{43}{54}x^3+\frac{41}{96}x^2-\frac{1}{8}x+\frac{1}{64}$?

Ans. $\frac{1}{2}x^2-\frac{2}{3}x+\frac{1}{4}$

52. Required the square root of a^3+2b.

Result, $a+\frac{b}{a}-\frac{b^2}{2a^2}+\frac{b^3}{2a^5}-\frac{5b^4}{8a^7}$

Section IV.

EQUATIONS.

41. An equation is an algebraic expression, indicating that one quantity, or combination of quantities, is equal

to another. The quantity, or combination, which stands on either side of the sign of equality, is called a member of the equation.

Thus, $ax^2+bx=ab+cd$, is an equation, of which ax^2+bx is the first member, and $ab+cd$ the second.

42. In the application of algebra to the solution of problems, the quantities concerned, both those which are given, and those which are required, are usually represented by symbols, and the conditions of the problem translated into the language of algebra. One or more equations are thus formed, including unknown as well as known quantities.

43. The number of independent equations required for the solution of a problem, is equal to the number of unknown quantities employed.

44. A *simple equation* is one which includes only the first or single power of the unknown quantity or quantities, as $ax+b=cx-d$.

45. A *quadratic equation*, is one which includes the square of the unknown quantity. An equation which contains no power of the unknown quantity but the square, is called a *simple quadratic*, as $ax^2=b$; but, when the square and simple power of the unknown quantity are both included, it is called an *adfected quadratic equation;* as $x^2-ax=bc$.

46. A *cubic equation*, is one which contains the cube of the unknown quantity; as $x^3+ax^2+bx=c$.

47. A *biquadratic equation*, is one which contains the fourth power of the unknown quantity; as $x^4+a^2x^2+c^3x=b^2d^2$.

48. Equations, in general, are said to be of as many dimensions as there are units in the index of the highest power of the unknown quantity.

49. In general, an equation which contains but one power of the unknown quantity is called a *pure equation;* thus, $x^2=b+c$, is called a pure quadratic equation; $x^3=b$, a pure cubic equation, &c.

Great part of the business of algebra consists in deducing, from given equations, the values of the unknown quantities which they contain. This is effected by making correspondent changes in both the members of the equation, so as to obtain, eventually, the unknown quantity on one side of the sign of equality, and known quantities on the other.

SIMPLE EQUATIONS,

Containing one unknown quantity.

To facilitate the solution of equations generally, the following principles or modes of operation must be observed.

Mode 1.

50. Any quantity may be transposed, that is, removed from one member of the equation to the other, by changing its sign.*

EXAMPLES.

Given $7x+5a=6x+3b$ to find x.

By transposing $5a$ and $6x$, we have $7x-6x=3b-5a$, or $x=3b-5a$.

2. Given $8y-3=7y+5$ to find y. Result, $y=8$.

3. Given $15x-6=14x+20$, to find x. Result, $x=26$.

4. Given $20-13y=45-14y+4$, to find y. Result, $y=29$.

5. Given $8x+15=15+9x-17$, to find x. Result, $x=17$.

* Hence, if any term be contained in both members of the equation with the same sign, it may be taken from both; and the signs of all the terms, may be changed, without destroying the equation.

6. Given $25x-b=24x+2b-c$, to find x.

Result, $x=3b-c$.

7. Given $3b-9x=7b-8x-4d$, to find x.

Result, $x=4d-4b$.

Mode 2.

51. The co-efficient of an unknown quantity may be taken away, by dividing every term of the equation by it.

EXAMPLES.

Given $ax+ba=ad$, to find x.

By dividing by a, we have $x+b=d$; hence by transposition, $x=d-b$.

2. Given $5x=25$, to find x. Result, $x=5$.

3. Given $30x=180$, to find x. Result, $x=6$.

4. Given $6x+17=44-3x$, to find x. Result, $x=3$.

5. Given $3ax-4ab=2ax-6ac$, to find x.

Result, $x=4b-6c$.

6. Given $3x^2-10x=8x+x^2$, to find x.

Result, $x=9$.

7. Given $6y^3-8y^3=22y^2-3y^3$, to find y.

Result, $y=3\frac{1}{2}$.

Mode 3.

52. An equation may be cleared of fractions by multiplying all its terms by any common multiple of the denominators.

EXAMPLES.

Given $\frac{x}{2}-\frac{x}{3}+\frac{x}{5}=1\frac{5}{6}$, to find x.

Multiply by 30, and then we have $15x-10x+6x=55$; whence $11x=55$, and $x=5$.

2. Given $\frac{x}{2}+\frac{x}{3}=\frac{x}{4}+7$, to find x. Result, $x=12$.

3. Given $\frac{x-5}{4}+6x=\frac{284-x}{5}$, to find x.

Result, $x=9$.

4. Given $x+\frac{11-x}{3}=\frac{19-x}{2}$, to find x. Result, $x=5$.

5. Given $3x+\frac{2x+6}{5}=5+\frac{11x-37}{2}$, to find x.

Result, $x=7$.

Mode 4.

53. When the unknown quantity is under a radical sign, such sign may be taken away, by first transposing the terms, so as to leave the radical quantity alone on one side of the equation, and then involving each member of the equation to the proper power.

EXAMPLES.

Given $\sqrt{x+40}=10-\sqrt{x}$, to find x.

Squaring both sides, $x+40=100-20\sqrt{x}+x$.

By transposition, $20\sqrt{x}=60$, or $\sqrt{x}=3$.

Squaring both members, $x=9$.

2. Given $\sqrt{x+1}-2=3$, to find x. Result, $x=24$.

3. Given $\sqrt[3]{3x+4}+3=6$, to find x. Result, $x=7\frac{1}{3}$.

4. Given $\sqrt{x-16}=8-\sqrt{x}$, to find x.

Result, $x=25$.

5. Given $\sqrt{4a+x}=2\sqrt{b+x}-\sqrt{x}$, to find x.

Result, $x=\frac{b^2-2ab+a^2}{2a-b}$

6. Given $\sqrt{4a^2+x^2}=\sqrt[4]{4b^4+x^4}$, to find x.

Result, $x=\sqrt{\frac{b^4-4a^4}{2a^2}}$

7. Given $x+a+\sqrt{2ax+x^2}=b$, to find x.

Result, $x=\frac{b^2-2ab+a^2}{2b}$

MODE 5.

54. When the member of an equation, which contains the unknown quantity, is a complete square, cube, &c. the equation may be reduced by extracting the square root, cube root, &c. of both its members.

EXAMPLES.

Given $x^3-6x^2+12x-8=343$, to find x.

Extracting the cube root, $x-2=7$, or $x=9$.

2. Given $x^2+4x+4=81$, to find x. Result, $x=7$.

3. Given $x^2+x+\frac{1}{4}=\frac{81}{4}$, to find x. Result, $x=4$.

4. Given $x^3+3ax^2+3a^2x+a^3=b^3$, to find x.

Result, $x=b-a$.

5. Given $9x^2+17=98$, to find x. Result, $x=3$.

6. Given $x^4-4x^3+6x^2-4x+1=256$, to find x.

Result, $x=5$.

7. Given $3x^5=96$, to find x. Result, $x=2$.

PROMISCUOUS EXAMPLES.

1. Given $\frac{1}{2}x-\frac{1}{4}x+\frac{1}{5}x=\frac{1}{2}$, to find x. Result, $x=1\frac{1}{9}$

2. Given $\sqrt{x}+\sqrt{2a+x}=\frac{4a}{\sqrt{2a+x}}$, to find x.

Result, $x=\frac{2}{3}a$.

3. Given $\frac{a}{1+2x}+\frac{a}{1-2x}=2b$, to find x.

Result, $x=\frac{1}{2}\sqrt{\frac{b-a}{b}}$

4. Given $a+x=\sqrt{a^2+x\sqrt{4b^2+x^2}}$, to find x.

Result, $x=\frac{b^2}{a}-a$.

5. Given $2+\sqrt{3x}=\sqrt{4+5x}$, to find x.

Result, $x=12$.

6. Given $a-x=\frac{x^2}{a-x}$, to find x. Result, $x=\frac{a}{2}$

7. Given $\sqrt{a+x}+\sqrt{a-x}=\sqrt{ax}$, to find x.

Result, $x=\frac{4a^2}{a^2+4}$

8. Given $\frac{a}{1+x}+\frac{a}{1-x}=b$ to find x. Result, $x=\sqrt{\frac{b-2a}{b}}$

9. Given $\frac{7x+16}{21}-\frac{x+8}{4x-11}=\frac{x}{3}$, to find x.

Result, $x=8$.

10. Given $\frac{4x+3}{9}+\frac{7x-29}{5x-12}=\frac{8x+19}{18}$, to find x.

Result, $x=6$.

11. Given $\frac{5x+4}{2}:\frac{18-x}{5}::7:2$ to find x.*

Result, $x=3\frac{5}{16}$.

12. Given $\sqrt{5+x}+\sqrt{x}=\frac{15}{\sqrt{5+x}}$, to find x.

Result, $x=4$.

13. Given $x^4+2x^3+1=4225$, to find x. Result, $x=8$.

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* If four numbers be proportional, the produet of the extremes is equal to that of the means; hence, an analogy is readily converted into an equation. Vide Article 62.

14. Given $x^4-9x^3+27x^2-27=2197$, to find x.

Result, $x=4$.

15. Given $x^3-17=130-2x^2$, to find x.

Result, $x=7$.

16. Given $x^3-100=412-7x^2$, to find x.

Result, $x=4$.

Section V.

SIMPLE EQUATIONS.

Containing two unknown quantities.

Rule 1.

55. Multiply or divide one or both equations, by such numbers or quantities as will make the co-efficients of one of the unknown quantities the same in both equations. Then, take the sum or difference of these equations, according as the signs of the corresponding terms are different, or the same; the result will be a new equation, containing but one unknown quantity.* The value of the remaining unknown quantity may then be determined by the methods in the last section.

EXAMPLES.

Given $4x+y=34$, and $4y+x=16$, to find the values of x and y.

First, to eliminate x, we multiply the second equation by 4; whence, $16y+4x=64$. From this take the first equation and $15y=30$; whence $y=2$.

The value of y being now obtained, we find $x=16-$

* When, by this or any other process, a quantity or letter is caused to vanish from the equations, it is said to be *eliminated*.

$4y=16-8=8$; or to eliminate x, divide the first equation by 4, and we shall have $x+\frac{1}{4}y=\frac{34}{4}$, which subtracted from the second equation, leaves $\frac{15}{4}y=\frac{30}{4}$ whence $y=2$, as before.

2. Given $ax+by=m$, $cx-dy=n$, to determine the values of x and y.

To eliminate x, multiply the first equation by c, and the second by a, whence $acx+bcy=cm$, and $acx-ady=an$; by subtraction $bcy+ady=cm-an$; and by dividing by $bc+ad$, $y=\frac{cm-an}{bc+ad}$.

Or, to eliminate y, multiplying the equations by d and b respectively, $adx+bdy=dm$, $bcx-bdy=bn$; adding these equations, $adx+bcx=dm+bn$; whence $x=\frac{dm+bn}{ad+bc}$.

3. Given $\begin{cases}5x+7y=50\\8x+3y=39,\end{cases}$ to find the values of x and y.

Result, $x=3$, $y=5$.

4. Given $\begin{cases}15x+34y=241,\\18x-17y=58,\end{cases}$ to find the values of x and y.

Result, $x=7$, $y=4$.

5. Given $\begin{cases}6x-4y=34,\\9y-2x=27.\end{cases}$ Required the values of x and y.

Ans. $x=9$, $y=5$.

6. Given $\begin{cases}11x+3y=100,\\4x-7y=4,\end{cases}$ to find the values of x and y.

Result, $x=8$, $y=4$.

7. Given $\begin{cases}\frac{x}{2}+\frac{y}{3}=7,\\\frac{x}{3}+\frac{y}{2}=8,\end{cases}$ to find x and y.

Result, $x=6$, $y=12$.

8. Given $\begin{cases} \frac{x}{15}-\frac{y}{20}=1, \\ \frac{y}{10}-\frac{x}{30}=1, \end{cases}$ to determine x and y.

Result, $x=30$, $y=20$.

RULE 2.

56. Find, from each equation, the value of one unknown quantity in terms of the other quantities.

Assume the values thus obtained, as the different members of a new equation. The value of the unknown quantity contained in this equation may then be determined as in the last section.

EXAMPLES.

Given $\begin{cases} 7x+\frac{y}{7}=51, \\ 9x-\frac{y}{8}=61\frac{1}{4}, \end{cases}$ to find the values of x and y.

From the first equation $y=357-49x$.

From the second, $y=72x-490$,

Hence, $72x-490=357-49x$,

$$72x+49x=357+490,$$

$$x=\frac{847}{121}=7,$$

$$y=357-49x=357-343=14.$$

2. Given $\begin{cases} \frac{x}{7}+7y=99, \\ 7x+\frac{y}{7}=51. \end{cases}$ Required the values of x and y.

Ans. $x=7$, $y=14$.

3. Given $\begin{cases} \frac{x+2}{3}+8y=31, \\ \frac{y+5}{4}+10x=192. \end{cases}$ Required the values of x and y.

Ans. $x=19$, $y=3$.

4. Given $\begin{cases} \frac{x}{5}+\frac{y}{9}=6, \\ \frac{x}{7}-\frac{y}{21}=2. \end{cases}$ Quere the values of x and y?

Ans. $x=20$, $y=18$.

5. Given $\begin{cases} \frac{2x}{3}+\frac{3y}{5}=22, \\ \frac{3x}{5}+\frac{2y}{3}=\frac{67}{3}. \end{cases}$ Quere the values of x and y?

Ans. $x=15$, $y=20$.

6. Given $\begin{cases} x-2y=d \\ x:y::a:b. \end{cases}$ Quere the values of x and y?

Ans. $x=\frac{ad}{a-2b}$, $y=\frac{bd}{a-2b}$

7. Given $\begin{cases} \frac{x-2y}{4}+\frac{x}{5}=\frac{3x-y}{5}-5\frac{2}{7}, \\ \frac{x+y}{4}+\frac{x-y}{7}=8\frac{5}{7}. \end{cases}$ Quere the values of x and y?

Ans. $x=20$, $y=8$.

RULE 3.

57. Find, in either equation, the value of one unknown quantity, and substitute the value thus found in the other equation; whence will arise a new equation, containing but one unknown quantity, whose value may be determined as already taught.

EXAMPLES.

Given $\begin{cases} x+\frac{y}{2}=12 \\ 7x-3y=58 \end{cases}$ to determine x and y.

From the first equation $x=12-\frac{y}{2}$. This value being

substituted for x in the second equation, we have

$$84-\frac{7y}{2}-3y=58.$$

Multiplying by -2, $7y+6y-168=-116$.

Whence $y=\frac{168-116}{13}=4$,

$$x=12-\frac{y}{2}=12-2=10.$$

2. Given $\begin{cases} 5x-3y=150, \\ 10x+15y=825. \end{cases}$ Quere the values of x and y?

Ans. $x=45$, $y=25$.

3. Given $\begin{cases} \frac{x}{3}-\frac{y}{5}=5, \\ \frac{x}{5}+y=31, \text{ to find } x \text{ and } y. \end{cases}$

Result, $x=30$, $y=25$.

4. Given $\begin{cases} x+y=24, \\ x:y::5:3, \text{ to find } x \text{ and } y. \end{cases}$

Result, $x=15$, $y=9$.

5. Given $\begin{cases} \frac{x}{3}+3y=21, \\ \frac{y}{3}+3x=29, \text{ to find } x \text{ and } y. \end{cases}$

Result, $x=9$, $y=6$.

Rule 4.

58. Multiply one of the equations by an indefinite quantity, and to the equation thus formed, add the other given equation.

Assume the sum of the co-efficients of one required quantity equal to 0, and thence determine the value of the assumed multiplier. The new equation will then contain but one unknown quantity, which may be found as before.

EXAMPLES.

Given $\left\{\begin{matrix}7x-9y=59\\3x+8y=132\end{matrix}\right\}$ to find the values of x and y.

Multiply the second equation by the indefinite quantity m, and we shall have $3mx+8my=132m$.

Adding this equation to the first,

$$3mx+7x+8my-9y=132m+59.$$

Assume $3m+7=0$, Whence $m=-\frac{7}{3}$,

$$\text{And } y=\frac{132m+59}{8m-9}=\frac{-308+59}{-\frac{56}{3}-9}=\frac{924-177}{56+27}=9.$$

$$\text{Hence } x=\frac{132-8y}{3}=\frac{132-72}{3}=\frac{60}{3}=20.$$

Or assuming $8m-9=0$, $m=\frac{9}{8}$

$$\text{And } x=\frac{132m+59}{3m+7}=\frac{\frac{1188}{8}+59}{\frac{27}{8}+7}=\frac{1188+472}{27+56}=20.$$

2. Given $\left\{\begin{matrix}15x-17y=12,\\17x-10y=97,\end{matrix}\right.$ to find x and y.

Result, $x=11$, $y=9$.

3. Given $\left\{\begin{matrix}3x+7y=79,\\2y-\frac{1}{2}x=9,\end{matrix}\right.$ to find x and y.

Result, $x=10$, $y=7$.

4. Given $\left\{\begin{matrix}8x-5y=60,\\10y-3x=75,\end{matrix}\right.$ to determine x and y.

Result, $x=15$, $y=12$.

5. Given $\begin{cases} \frac{x}{2}-12=\frac{y}{4}+13 \\ \frac{x+y}{5}+\frac{x}{3}+16=\frac{2x-y}{4}+27. \end{cases}$ Required the values of x and y? Result, $x=60$, $y=20$.

*Examples to exercise the foregoing rules.**

Given $\begin{cases} x-y=d, \\ xy=p, \end{cases}$ to find x and y.

Squaring the first equation, $x^2-2xy+y^2=d^2$.

Multiplying the second by 4, $4xy=4p$.

Taking the sum, $x^2+2xy+y^2=d^2+4p$.

Extracting the root, $x+y=\sqrt{d^2+4p}$, (*A*.)

Adding the first, $2x=d+\sqrt{d^2+4p}$, or $x=\frac{d+\sqrt{d^2+4p}}{2}$

Subtract first equation from equation *A*, and

$$2y=\sqrt{d^2+4p}-d, \text{ or } y=\frac{\sqrt{d^2+4p}-d}{2}.$$

2. Given $\begin{cases} x-y=m, \\ x^2-y^2=n, \end{cases}$ to find x and y.

Dividing the second by the first, $x+y=\frac{n}{m}$.

Hence, adding and subtracting the first, $2x=\frac{n}{m}+m$ and $2y=\frac{n}{m}-m$.

Whence, $x=\frac{n+m^2}{2m}$ and $y=\frac{n-m^2}{2m}$.

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* In the solution of these and other similar problems, the ingenious student will find expedients which are not clearly indicated by any of the preceding rules, by which his labour may be frequently abridged.

3. Given $\begin{cases} x^2y-xy^2=a, \\ x^3-y^3=b, \end{cases}$ to find x and y.

Subtracting three times the first equation from the second, $x^3-3x^2y+3xy^2-y^3=b-3a$.

Hence, extracting the root, $x-y=\sqrt{b-3a}$, which put $=c$, $(A.)$

Divide the first equation by this, whence $xy=\frac{a}{c}$ $(B.)$

To the square of equation $(A,)$ add four times equation $(B,)$ and we shall have

$$x^2+2xy+y^2=c^2+\frac{4a}{c},$$

Whence by evolution, $x+y=\sqrt{(c^2+\frac{4a}{c})}$

Adding and subtracting equation A, we find

$$2x=\sqrt{(c^2+\frac{4a}{c})}+c, \text{ and } 2y=\sqrt{(c^2+\frac{4a}{c})}-c;$$

$$\text{Or } x=\frac{\sqrt{(c^2+\frac{4a}{c})}+c}{2}, \text{ and } y=\frac{\sqrt{(c^2+\frac{4a}{c})}-c}{2}$$

4. Given $\begin{cases} x+y=a, \\ x^3+y^3=b, \end{cases}$ to find x and y.

From the first $x^3+3x^2y+3xy^2+y^3=a^3$.

By subtraction, $3x^2y+3xy^2=a^3-b$.

Dividing by three times the first, $xy=\frac{a^3-b}{3a}$, which put $=c$.

Subtracting four times this equation from the square of the first, $x^2-2xy+y^2=a^2-4c$.

Hence, $x-y=\sqrt{a^2-4c}$.

Whence, $x=\frac{a+\sqrt{a^2-4c}}{2}$; and $y=\frac{a-\sqrt{a^2-4c}}{2}$

Otherwise, dividing the second equation by the first,

$$x^2-xy+y^2=\frac{b}{a}.$$

Hence, $4x^2-4xy+4y^2=\frac{4b}{a}$

Squaring the first equation, $x^2+2xy+y^2=a^2$.
Subtracting the latter from the former, we have

$$3x^2-6xy+3y^2=\frac{4b-a^2}{a};$$

Consequently, $x^2-2xy+y^2=\frac{4b-a^2}{3a}$,

And, by extracting the root, $x-y=\sqrt{\frac{4b-a^2}{3a}}$

Whence, $x=\frac{a+\sqrt{\frac{4b-a^2}{3a}}}{2}$ and $y=\frac{a-\sqrt{\frac{4b-a^2}{3a}}}{2}$

If we put $\frac{a^2-b}{3a}=c$, the values of x and y will be found the same as before.

5. Given $\begin{cases} x+y=s, \\ x-y=d, \end{cases}$ to find x and y in terms of s and d.

Result, $x=\frac{s+d}{2}$, $y=\frac{s-d}{2}$

6. Given $\begin{cases} x+y=s, \\ x^2-y^2=d, \end{cases}$ to find as before.

$x=\frac{s^2+d}{2s}$, $y=\frac{s^2-d}{2s}$

7. Given $\begin{cases} x+y=6, \\ xy=8. \end{cases}$ Quere the values of x and y.

Ans. $x=4$, $y=2$.

8. Given $\begin{cases} x:y::3:2, \\ x^2-y^2=20, \end{cases}$ to find x and y.

Result, $x=6$, $y=4$.

9. Given $\begin{cases} \frac{x-2}{5}-\frac{10-x}{3}=\frac{y-10}{4}, \\ \frac{2y+x}{3}-\frac{2x+y}{8}=\frac{x+17}{4}. \end{cases}$ Required the numerical values of x and y.

Ans. $x=7$, $y=10$.

10. Given $\begin{cases} x+1 : y-1 :: 5 : 2, \\ \frac{2x}{3} - \frac{5-y}{2} = \frac{41}{12} - \frac{2x-1}{4}, \end{cases}$ to find x and y.

Result, $x=4$, $y=3$.

11. Given $\begin{cases} \frac{y}{9} - \frac{4x-1}{18} = 1 - \frac{4+y}{3} + \frac{x-y}{6}, \\ x : 3y :: 4 : 7, \text{ to find } x \text{ and } y. \end{cases}$

Result, $x=12$, $y=7$.

12. Given $\begin{cases} \frac{4}{x} + \frac{5}{y} = \frac{9}{y} - 1, \\ \frac{5}{x} + \frac{4}{y} = \frac{7}{x} + \frac{3}{2}, \text{ to find } x \text{ and } y. \end{cases}$

Result, $x=4$, $y=2$

13. Given $\begin{cases} x^2y+xy^2=120, \\ x^3+y^3=152. \end{cases}$ What are the values of x and y?

Ans. $x=5$, $y=3$.

14. Given $\begin{cases} x+y=12, \\ x^3+y^3=468, \text{ to find } x \text{ and } y. \end{cases}$

Result, $x=7$, $y=5$.

15. Given $\begin{cases} x^2+xy=112, \\ xy+y^2=84, \text{ to find } x \text{ and } y. \end{cases}$

Result, $x=8$, $y=6$.

16.* Given $\begin{cases} ax+by=m, \\ dx+ey=n, \text{ to find } x \text{ and } y. \end{cases}$

Result, $x=\frac{em-bn}{ae-db}$, $y=\frac{an-md}{ae-db}$

17. Given $\begin{cases} x^4-y^4=12240, \\ x^2+y^2=170, \text{ to find } x \text{ and } y. \end{cases}$

Result, $x=11$, $y=7$.

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* When we have two equations, and two unknown quantities, neither of which rises above the first power, the required values may be obtained, from the results of the 16th example, by simple substitution.

18. Given $\begin{cases} x-y=2, \\ x^2-y^2=98, \end{cases}$ to determine x and y.

Result, $x=5$, $y=3$.

Section VI.

SIMPLE EQUATIONS,

Containing three or more unknown quantities.

59. The unknown quantities may be eliminated one after another, by the first three rules laid down in the last section. Or multiply each given equation, except one, by an indefinite quantity; add the remaining given equation, and all the new equations together; and assume the sum of the co-efficients of each required quantity, except one equal to 0; whence, as many equations as there are assumed multipliers will be formed: and thence the values of those multipliers may be determined, and consequently, the last unknown quantity.

There are other methods, which may often be advantageously applied, but which can be learned only from practice.

EXAMPLES.

Given $\left.\begin{cases} x+y+z=29, \\ x+2y+3z=62, \\ \frac{x}{2}+\frac{1}{3}y+\frac{1}{4}z=10. \end{cases}\right\}$ to find the values of x, y, and z.

Multiply the last equation by 2, $x+\frac{2}{3}y+\frac{1}{2}z=20$.

Subtract this from the first equation, $\frac{1}{3}y+\frac{1}{2}z=9$.

Multiply by 3, and $y+\frac{3}{2}z=27$.

Subtract the first equation from the second, and

$$y+2z=33.$$

From this equation subtract the former, and we find

$$\frac{1}{2}z=6 \text{ or } z=12.$$

Take double the last equation from the last but two, and $y=9$.

Lastly, the sum of these last equations subtracted from the first, leaves $x=8$.

Otherwise.—From the given equations,

$$x=29-y-z.$$

$$x=62-2y-3z.$$

$$x=20-\frac{2}{3}y-\frac{1}{2}z.$$

Whence, $29-y-z=62-2y-3z.$

And $20-\frac{2}{3}y-\frac{1}{2}z=29-y-z.$

From the former of these, $y=33-2z$,

And from the latter, $y=27-\frac{3}{2}z.$

Therefore, $27-\frac{3}{2}z=33-2z.$

Or, $54-3z=66-4z$; $\therefore z=12$.*

But $y=33-2z=9.$

And $x=29-y-z=8.$

* The sign $\therefore$ is used in the place of the word *therefore*.

Or multiplying the second and third equations by m and n respectively, $mx+2my+3mz=62m$,

$$\frac{1}{2}nx+\frac{1}{3}ny+\frac{1}{4}nz=10n.$$

Hence, by adding these equations to the first,

$$x+mx+\frac{1}{2}nx+y+2my+\frac{1}{3}ny+z+3mz+\frac{1}{4}nz=29+62m+10n.$$

Assume $1+2m+\frac{1}{3}n=0$, and $1+3m+\frac{1}{4}n=0$.

From the former of these equations $\frac{3}{2}+3m+\frac{1}{2}n=0$.

$$\therefore \frac{1}{2}+\frac{1}{4}n=0, \text{ or } n=-2.$$

$$\text{Hence, } m=-\frac{1}{6}.$$

$$\text{Whence, } x=\frac{29+62m+10n}{1+m+\frac{1}{2}n}=\frac{29-\frac{62}{6}-20}{1-\frac{1}{6}-1}=8.$$

Again, making $1+m+\frac{1}{2}n=0$, and $1+3m+\frac{1}{4}n=0$,

We have $n=-\frac{8}{5}$, and $m=-\frac{1}{5}$.

$$\text{Consequently, } y=\frac{29+62m+10n}{1+2m+\frac{1}{3}n}=9.$$

Hence, $z=12$.

2. Given $\begin{cases} x+y+z=53, \\ x+2y+3z=105, \\ x+3y+4z=134, \end{cases}$ to find x, y, and z.

Subtract the first from the second, $y+2z=52$.

Second from third, $y+z=29$.

Subtracting the last, $z=23$.

$\therefore y=6$.

And $x=24$.

3. Given $\begin{cases} x+y=20, \\ x+z=24, \\ y+z=30, \end{cases}$ to find x, y, and z.

Result, $x=7$, $y=13$, $z=17$.

4. Given $\begin{cases} 7x+5y+2z=79, \\ 8x+7y+9z=122, \\ x+4y+5z=55, \end{cases}$ to determine the values of x, y, and z.

Result, $x=4$, $y=9$, $z=3$.

5. Given $\begin{cases} x-y=2, \\ x-z=3, \\ y+z=9, \end{cases}$ to find x, y, and z.

Result, $x=7$, $y=5$, $z=4$.

6. Given $\begin{cases} 2x+3y+4z=38, \\ 5x+7y+11z=98, \\ 7x+9y+15z=132, \end{cases}$ to find x, y, and z.

Result, $x=3$, $y=4$, $z=5$.

7. Given $\begin{cases} ax+by+cz=m, \\ dx+ey+fz=n, \\ gx+hy+kz=p, \end{cases}$ to find x, y, and z,

Result,* $\begin{cases} x=\dfrac{kem-hfm+hcn-bkn+bfp-ecp}{aek-ahf+dhc-dbk+gbf-gec} \\ y=\dfrac{akn-afp+dcp-dkm+gfm-gcn}{aek-ahf+dhc-dbk+gbf-gec} \\ z=\dfrac{aep-ahn+dhm-dbp+gbn-gem}{aek-ahf+dhc-dbk+gbf-gec} \end{cases}$

* From these general results the values of x, y, z, in the preceding examples, may be found by simple substitution.

8. Given $\begin{cases} v+x+y=30, \\ v+x+z=31, \\ v+y+z=32, \\ x+y+z=33, \text{ to find } v, x, y, \text{ and } z. \end{cases}$

Result, $v=9$, $x=10$, $y=11$, $z=12$.

9. Given $\begin{cases} \dfrac{4x+3y+z}{10}-\dfrac{2y+2z-x+1}{15}=5+\dfrac{x-z-5}{5}, \\ \dfrac{9x+5y-2z}{12}-\dfrac{2x+y-3z}{4}=\dfrac{7y+z+3}{11}+\dfrac{1}{6}, \\ \dfrac{5y+3z}{4}-\dfrac{2x+3y-z}{12}+2z=y-1+\dfrac{3x+2y+7}{6}. \end{cases}$

Required the values of x, y, and z.

Ans. $x=9$, $y=7$, $z=3$.

10. Given $\begin{cases} 3v+4x+5y+6z=102, \\ 4v-6x+4y-3z=6, \\ 2v+3x-5y+4z=11, \\ 6v+5x-7y+z=7, \text{ to find } v, x, y, \text{ and } z. \end{cases}$

Result, $v=5$, $x=4$, $y=7$, $z=6$.

Examples producing simple equations.

Required to find two numbers whose difference shall be 4, and the difference of the squares 64.

Let $x=$ the greater number, and $y=$ the less,

Then, by the question, $\begin{cases} x-y=4, \\ x^2-y^2=64. \end{cases}$

Dividing the 2d equation by the 1st, $x+y=16$.

Taking the sum and difference of this equation and the first, $2x=20$, and $2y=12$.

$\therefore x=10$, and $y=6$.

Otherwise. Let $x=$ the less number, then $x+4=$the greater:

Whence, by the question, $\overline{x+4}\,]^2-x^2=64$,

Or $8x+16=64$.

$\therefore$ by transposition and division, $x=6$, and $x+4=10$.

2. A person having bought three loads of grain, the first, consisting of 30 bushels of rye, 20 of barley, and 10 of wheat, for 230 francs; the second, containing 15 bushels of rye, 6 of barley, and 12 of wheat, for 138 francs; the third, consisting of 10 bushels of rye, 5 of barley, and 4 of wheat, for 75 francs; required the price per bushel of each of these kinds of grain.

Suppose $\left\{\begin{array}{l}\text{a bushel of rye cost } x,\\ \text{a bushel of barley, } y,\\ \text{and a bushel of wheat, } z.\end{array}\right\}$ francs.

Then by the question, $\left\{\begin{array}{l}30x+20y+10z=230, (A.)\\ 15x+6y+12z=138,\\ 10x+5y+4z=75,\end{array}\right.$

Multiplying the 2d equation by 2, and the 3d by 3,

$$30x+12y+24z=276,$$

$$30x+15y+12z=225. (B.)$$

Taking the difference of these equations,

$$3y-12z=-51. (C.)$$

Take the difference of equations $(A.)$ and $(B.)$ and

$$5y-2z=5.$$

Multiplying this by 6, $30y-12z=30$.

And subtracting equation $(C.)$

$$27y=81. \quad \text{Whence, } y=3.$$

$$\therefore z=\frac{15-5}{2}=5, \text{ and } x=\frac{23-6-5}{3}=4.$$

3. Required to find two numbers, such that their difference multiplied by their product, shall be 308, and the difference of their cubes 988.

Let x= the greater, and y= the less.

Then from the question,

$$(x-y).xy, \text{ or } x^2y-xy^2=308.$$

$$\text{And } x^3-y^3=988.$$

From the latter taking thrice the former,

$$\text{We have } x^3-3x^2y+3xy^2-y^3=64.$$

And taking the cube root, $x-y=4$. (*A.*)

Dividing the sum of the two primary equations by this last,

$$\frac{x^3+x^2y-xy^2-y^3}{x-y}=x^2+2xy+y^2=324.$$

Hence, by extracting the square root, $x+y=18$. (*B.*)

Taking the sum and difference of equations (*A.*) and (*B.*)

$$x=\frac{18+4}{2}=11, \text{ and } y=\frac{18-4}{2}=7.$$

4. The sum of two numbers multiplied by their difference is 20, and the difference of their 4th powers 1040; what are the numbers?

Let x= the greater, y= the less.

Then from the *data*, $\begin{cases} \overline{x+y}.\overline{x-y}=x^2-y^2=20. \\ \text{And } x^4-y^4=1040. \end{cases}$

Dividing the latter by the former, $x^2+y^2=52$.

Taking the sum and difference of the first and last equations, $2x^2=72$, and $2y^2=32$.

$$\therefore x=\sqrt{36}=6.$$

$$\text{And } y=\sqrt{16}=4.$$

5. The difference of two numbers is 5, and the difference of their cubes 1685; what are the numbers?

Let x= the greater, y= the less.

Then *per data*, $\begin{cases} x-y=5. \\ x^3-y^3=1685. \end{cases}$

By division, $\frac{x^3-y^3}{x-y}=x^2+xy+y^2=337$.

Squaring the first equation, $x^2-2xy+y^2=25$.

By subtraction, $3xy=312$.

Adding $\frac{4}{3}$ of this equation to the preceding,

$$x^2+2xy+y^2=441.$$

Extracting the square root, $x+y=21$.

From this and the first equation, $x=13$, and $y=8$.

6. Required to find two numbers, such that half the first added to a third of the second shall make 9, and a fourth of the first, added to a fifth of the second, shall make 5. Ans. 8 and 15.

7. A testator bequeaths to his widow $\frac{2}{3}$ of his estate, to his son $\frac{1}{3}$, and the remainder, which is found to be 756 dollars, to his daughter; what was the estate left? Ans. 2835 dollars.

8. There are two numbers in the ratio of 3 to 5, and their sum is one fourth of the difference of their squares; what are the numbers? Ans. 6 and 10.

9. The ages of a man and his wife, at the time of marriage, were in the ratio of 7 to 6, and at the end of 30 years, they are found to be as 17 to 16. Quere their ages at the former period? Ans. 21 and 18.

10. A post being $\frac{1}{4}$ of its length in the earth, $\frac{1}{3}$ in the water, and 10 feet above the water; what was its whole length? Ans. 24 feet.

11. Divide 1100 dollars among A. B. and C., so that B. shall have 100 dollars more than A., and C. 150 more than B.; what is the share of each?
Ans. A. 250, B. 350, C. 500.

12. There are 21 persons, whose ages form an equidifferent series; the age of the eldest is 4 times that of the youngest, and the sum of all their ages is 525 years; how old is the youngest? Ans. 10 years.

13. At a certain election, 1296 persons voted, and the successful candidate had a majority of 120; how many voted for each? Ans. 708 for one, and 588 for the other.

14. A servant agreed to serve his master for £8 a year and his livery, but was turned away at the end of 7 months, and received only £2 13*s*. 4*d*. besides his livery; what was the price of his livery? Ans. £4 16*s*.

15. A servant having eloped from his master, travels 14 hours in the day, at the rate of $3\frac{1}{2}$ miles an hour; at the end of two days, a courier is sent in pursuit, who rides 9 hours in the day, at the rate of 7 miles an hour; in what time, and at what distance will he overtake him?

Ans. 7 days, and 441 miles.

16. The sun's mean daily motion in the ecliptic is 59′ 8″, and that of the moon 13° 10′ 35″, what is the time of a mean synodic revolution of the moon, viz. a revolution from conjunction with the sun to conjunction again?

Ans. 29 days, 12 hours, 44 minutes nearly.

17. A farmer, having hired a labourer, on condition, that for every day he wrought he should receive 50 cents, and for every day he was idle he should forfeit 20 cents, finds at the end of 420 days, that neither is indebted to the other; how many days did he labour? Ans. 120.

18. A certain number, consisting of two places of figures, is equal to the difference of the squares of its digits; and if 36 be added to it, the order of the digits will be inverted; what is the number? Ans. 48.

19. A vintner proposing to mix three sorts of wine, viz. at 65 cents per gallon, at 45 cents, and at 35 cents; the number of gallons at 65 cents, to the number at 35 cents, being as 5 to 3, so as to compose a hogshead worth 50 cents per gallon. Quere the number of each?

Ans. $22\frac{1}{2}$ at 65, 27 at 45, and $13\frac{1}{2}$ at 35.

20. There are 4 equidifferent numbers, whose sum is 56, and the sum of whose squares is 864; what are the numbers? Ans. 8, 12, 16, and 20.

21. If 38 federal dollars and 11 English crowns be

given for 271 French francs, and 76 dollars and 33 crowns for 610 francs; at how many francs were the dollar and crown respectively estimated?

Ans. the dollar $5\frac{13}{38}$, the crown $6\frac{2}{11}$.

22. There is a number consisting of three digits in arithmetical progression, whose sum is 21; and if to the number 396 be added, the sum will be expressed by the same digits in an inverted order; what is the number?

Ans. 579.

23. Required to find four numbers, such that the continued product of the 1st, 2d, and 3d, shall be 24; the product of the 1st, 2d, and 4th, 30; of the 1st, 3d, and 4th, 40; and of the 2d, 3d, and 4th, 60. Result, 2, 3, 4, 5.

24. It is required to find three numbers, such that the first multiplied by the sum of the other two shall be 96; the second multiplied by the sum of the other two shall be 105; and the third multiplied by the sum of the former two shall be 117. The numbers are, 6, 7, and 9.

25. What fraction is that, to the numerator of which if 1 be added, the value will be $\frac{1}{3}$, but if to the denominator 1 be added, the value will be $\frac{1}{4}$? Ans. $\frac{4}{15}$.

26. A person bought a chaise, horse, and harness, for 150 dollars; the price of the chaise was twice the price of the harness, and the price of the horse as much and half as much as the price of the chaise and harness; what was the cost of each?

Ans. harness 20 dollars, chaise 40, horse 90.

27. Two persons, A. and B. have the same income: A. saves $\frac{1}{7}$ of his yearly; but B., by spending as much in 3 years as A. does in 4, finds himself, at the end of five years, 200 dollars in debt; what was the income?

Ans. 600 dollars.

28. A. and B. put equal sums in trade: A. gained a sum equal to $\frac{1}{4}$ of his stock, and B. lost 450 dollars; when A's. money was double of B's.; what was the sum laid out by each? Ans. 1200 dollars.

29. Two persons comparing their ages, find them, at

present, in the ratio of 7 to 5, but that 30 years ago, they were in the ratio of 2 to 1; what are their ages?

Ans. 70 and 50.

30. A. and B. began trade with equal sums of money. In the first year A. gained £40, and B. lost £40; but in the second, A. lost one third of what he then had, and B. gained a sum which was £40 less than twice what A. had lost; when it appeared, that B. had twice as much as A.; what sum had each of them at first? Ans. £320.

31. A farmer sold 96 loads of hay to two persons. To the first one half, and to the second one fourth of what his stack contained; how many loads were in the stack?

Ans. 128.

32. If 116 be divided into four parts in such manner, that the first being increased by 5, the second diminished by 4, the third multiplied by 3, and the fourth divided by 2, the results will all be the same; what are the parts?

Ans. 22, 31, 9, and 54.

33. A shepherd in time of war, was plundered by a party, who took from him $\frac{1}{4}$ of his flock, and $\frac{1}{4}$ of a sheep more; another party took from him $\frac{1}{3}$ of what he had left, and $\frac{1}{3}$ of a sheep more; afterwards, a third party took $\frac{1}{2}$ of what remained, and $\frac{1}{2}$ a sheep more, when he had but 25 left; how many had he at first? Ans. 103.

34. A person has two horses and a gig; the gig is worth 150 dollars. When the first horse is attached to the gig, the value of the two is twice that of the second; but when the second horse is put to the gig, the value is three times that of the first horse; what were the horses respectively worth?

Ans. the first 90 dollars, the second 120.

35. When a company at a tavern came to pay their reckoning, they found, that if there had been three persons more, they would have had a shilling a piece less to pay; but if there had been two less, they would have had a shilling a piece more to pay; required the number of persons, and the quota of each?

Ans. 12 persons, quota of each 5*s*.

36. There are three equidifferent numbers, whose sum is 324, and the first is to the third as 5 to 7; what are the numbers? Ans. 90, 108, and 126.

37. A man and his wife usually drank a cask of beer in 12 days; but when the man was from home, it lasted the woman 30 days; how long would the man alone be in drinking it? Ans. 20 days.

38. A. and B. can perform a piece of work in 8 days, A. and C. in 9 days, and B. and C. in 10 days; how many days would they severally require to perform the same work? Ans. A. $14\frac{34}{49}$ days, B. $17\frac{23}{41}$, and C. $23\frac{7}{31}$.

39. The hypothenuse of a right angled triangle is 13, and the area 30; what are the other two sides?* Ans. 12 and 5.

40. There are four numbers in geometrical progression; the sum of the extremes is 18, and the sum of the means 12; what are the numbers? Ans. 2, 4, 8, and 16.

41. Required to find four numbers in geometrical progression, such that the difference of the means shall be 100, and the difference of the extremes 620.
The numbers are 5, 25, 125, and 625.

Section VII.

QUADRATIC EQUATIONS.

60. When an equation contains the square and simple power of an unknown quantity; or in general, two powers, one of whose indices is double the other, of an unknown quantity, whether that unknown quantity be simple or compound, it is called an *adfected quadratic equation*.

* To solve this problem, it must be recollected that the square of the hypothenuse is equal to the sum of the squares of the other two sides, and that the area is half the product of those sides.

Rule 1.

Arrange the terms, by transposition or otherwise, so that the highest power of the unknown quantity shall be contained in the first, and the other power in the second term on the left side of the equation; and the term or terms consisting of known quantities, on the other side.

When the highest power of the unknown quantity contains a co-efficient, divide the equation by that co-efficient. Then add, to each member of the equation, the square of half the co-efficient of the unknown quantity, (or the numerical part of the second term,) and the first member of the equation will be a complete square.

Extract the root and find the value of the unknown quantity by the former methods.

Rule 2.

61. Arrange the terms as before. When the highest power of the unknown quantity has a co-efficient, multiply the equation by four times that co-efficient, and to each member of the resulting equation, add the square of the co-efficient of the unknown quantity, contained in the primary equation; then the first member of the resulting equation will be a complete square.

Extract the root, and proceed by the former methods, to determine the value of the quantity sought.

EXAMPLES.

Given $2x^2-9x=266$, to find the value of x.

By rule 1.—Dividing by 2, $x^2-\frac{9}{2}x=133$.

Adding $\left(\frac{9}{4}\right)^2$ to each member, $x^2-\frac{9}{2}x+\frac{81}{16}=\frac{2209}{16}$

Extracting the root, $x-\frac{9}{4}=\frac{47}{4}$

$\therefore x=14$.

By rule 2.—Multiplying by 8, $16x^2-72x=2128$.

Adding 81 to each member, $16x^2-72x+81=2209$.

Extracting the square root, $4x-9=47$.

$$\therefore x=\frac{47+9}{4}=14, \text{ as before.}$$

2. Given $ax^2+bx=c$, to find the value of x.

By rule 1.—Dividing by a, $x^2+\frac{b}{a}x=\frac{c}{a}$.

Adding $\frac{b^2}{4a^2}$ to each, $x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2+4ac}{4a^2}$

By evolution, $x+\frac{b}{2a}=\frac{\sqrt{b^2+4ac}}{2a}$

By transposition, $x=\frac{\sqrt{b^2+4ac}-b}{2a}$

By rule 2.—Multiplying by $4a$, $4a^2x^2+4abx=4ac$.

Adding b^2 to each, $4a^2x^2+4abx+b^2=4ac+b^2$.

By evolution, $2ax+b=\sqrt{4ac+b^2}$.

By transposition and division, $x=\frac{\sqrt{4ac+b^2}-b}{2a}$

3. Given $2x^{\frac{2}{3}}+3x^{\frac{1}{3}}=2$, to find the value of x.

By rule 1. $x^{\frac{2}{3}}+\frac{3}{2}x^{\frac{1}{3}}=1$.

Comp. the sqr. $x^{\frac{2}{3}}+\frac{3}{2}x^{\frac{1}{3}}+\frac{9}{16}=\frac{25}{16}$

Evolution, $x^{\frac{1}{3}}+\frac{3}{4}=\frac{5}{4}$

$$\therefore x^{\frac{1}{3}} = \frac{5-3}{4} = \frac{1}{2}$$

Whence, $x = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

By rule 2. $16x^{\frac{2}{3}} + 24x^{\frac{1}{3}} = 16.$

Adding 9 to each, $16x^{\frac{2}{3}} + 24x^{\frac{1}{3}} + 9 = 25.$

By evolution, $4x^{\frac{1}{3}} + 3 = 5.$

By transp. and division, $x^{\frac{1}{3}} = \frac{1}{2}$

Therefore, $x = \frac{1}{8}$

4. Given $x^4 - 12x^3 + 44x^2 - 48x = 9009$, to determine x.

This equation is equivalent to the following:

$$x^4 - 12x^3 + 36x^2 + 8x^2 - 48x = 9009,$$

Or to this $(x^2 - 6x)^2 + 8.(x^2 - 6x) = 9009.$

In which, $x^2 - 6x$ may be considered as the unknown quantity.

Hence, proceeding as in the former examples,

$$(x^2 - 6x)^2 + 8.(x^2 - 6x) + 16 = 9025.$$

$$\therefore x^2 - 6x + 4 = 95, \text{ or } x^2 - 6x = 91.$$

Whence, $x^2 - 6x + 9 = 100.$

$$\therefore x - 3 = 10, \text{ or } x = 13.$$

5. Given $\surd(x - \frac{1}{x}) + \surd(1 - \frac{1}{x}) = x$, to find the value of x.

Multiplying by $\surd x$, $\surd\overline{x^2 - 1} + \surd\overline{x - 1} = x\surd x.$

Squaring this, $x^2-1+2\sqrt{x^3-x^2-x+1}+x-1=x^3$.

By transposing and changing the signs,

$$x^3-x^2-x+1-2\sqrt{x^3-x^2-x+1}=-1.$$

Here $\sqrt{x^3-x^2-x+1}$, may be considered as the unknown quantity.

Hence, adding 1 to each side,

$$x^3-x^2-x+1-2\sqrt{x^3-x^2-x+1}+1=0.$$

$$\sqrt{x^3-x^2-x+1}-1=0, \text{ or } \sqrt{x^3-x^2-x+1}=1.$$

$$\therefore x^3-x^2-x+1=1;$$

Whence, $x^2-x=1$.

$$x^2-x+\frac{1}{4}=\frac{5}{4}; x-\frac{1}{2}=\sqrt{\frac{5}{4}}=\frac{1}{2}\sqrt{5}.$$

$$\therefore x=\frac{1}{2}+\frac{1}{2}\sqrt{5}.$$

6. Given $3x^{2n}-2x^n+3=11$, to find x.

$$3x^{2n}-2x^n=8.$$

By rule 1. $x^{2n}-\frac{2}{3}x^n=\frac{8}{3}$

$$x^{2n}-\frac{2}{3}x^n+\frac{1}{9}=\frac{25}{9}$$

$$x^n-\frac{1}{3}=\frac{5}{3}, \text{ or } x=2^{\frac{1}{n}}.$$

By rule 2. $36x^{2n}-24x^n=96.$

$$36x^{2n}-24x^n+4=100.$$

$$6x^n-2=10.$$

$$x^n=\frac{12}{6}=2, \text{ or } x=\sqrt[n]{2}, \text{ as before.}$$

7. Given $x^{4n}-2x^{3n}+x^n=132$, to find the value of x.

This equation may be changed to

$$x^{4n}-2x^{3n}+x^{2n}-x^{2n}+x^n=132.$$

Which is $(x^{2n}-x^n)^2-(x^{2n}-x^n)=132$.

Complete the square,

$$(x^{2n}-x^n)^2-(x^{2n}-x^n)+\frac{1}{4}=132\tfrac{1}{4}=\frac{529}{4}$$

Extract the root, $x^{2n}-x^n-\frac{1}{2}=\frac{23}{2}$

$$\therefore x^{2n}-x^n=12.$$

Complete the square, $x^{2n}-x^n+\frac{1}{4}=12\frac{1}{4}=\frac{49}{4}$

Extract the root, $x^n-\frac{1}{2}=\frac{7}{2}$

$$\therefore x^n=4, \text{ or } x=\sqrt[n]{4}.$$

8. Given $x^2+12x=64$. Required the value of x.

Ans. $x=4$.

9. Given $7x^2+5x+4=82$, to find x. Ans. $x=3$.

10. Given $\frac{1}{2}x^2-\frac{1}{3}x+7\frac{3}{8}=8$, to find x.

Ans. $x=1\frac{1}{2}$.

11. Given $x^6+20x^3=224$, to find x. Ans. $x=2$.

12. Given $5x^2-4x+3=159$, to find x. Ans. $x=6$.

13. Given $6x+\frac{35-3x}{x}=44$, to find x.

Ans. $x=7$, or $\frac{5}{6}$.*

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* This example falls under the ambiguous case, which is explained in the next note.

14. Given $4x-\frac{14-x}{x+1}=14$, to find the value of x.

Ans. $x=4$.

15. Given $3x-\frac{1121-4x}{x}=2$, to find the value of x.

Ans. $x=19$.

16. Given $5x-\frac{3x-3}{x-3}=2x+\frac{3x-6}{2}$, to find x.

Ans. $x=4$.

62. The foregoing solutions have been effected by executing the whole process in each case; but it is more convenient in practice, as well as more elegant, to solve all such equations by a few general *formulæ*. In order to which, it is to be observed, that all adfected quadratic equations are reducible to one of the following forms.

$$\left.\begin{array}{l} 1.\ x^2+2ax=b. \\ 2.\ x^2-2ax=b. \\ 3.\ x^2-2ax=-b. \end{array}\right\}(A.)$$

These equations, when the squares are completed, become,

$$1.\ x^2+2ax+a^2=a^2+b$$

$$2.\ x^2-2ax+a^2=a^2+$$

$$3.\ x^2-2ax+a^2=a^2-b.$$

Hence, by evolution and transposition, we have

$$1.\ x=-a+\sqrt{a^2+b}.$$

$$2.\ x=a+\sqrt{a^2+b}.$$

$$3.\ x=a\pm\sqrt{a^2-b}.*$$

Adfected quadratic equations are sometimes expressed by one of the following forms.

$$\left.\begin{array}{l} 1.\ ax^2+bx=c. \\ 2.\ ax^2-bx=c. \\ 3.\ ax^2-bx=-c. \end{array}\right\}(B.)$$

* See note, page 77.

Multiplying these equations severally by $4a$, and adding b^2 to each member, they become,

1. $4a^2x^2+4abx+b^2=b^2+4ac.$
2. $4a^2x^2-4abx+b^2=b^2+4ac.$
3. $4a^2x^2-4abx+b^2=b^2-4ac.$

The roots being extracted, the values of x are easily found.

1. $x=\dfrac{-b+\sqrt{b^2+4ac}}{2a}$

2. $x=\dfrac{b+\sqrt{b^2+4ac}}{2a}$

3. $x=\dfrac{b\pm\sqrt{b^2-4ac}}{2a}$*

From these *formulæ*, any adfected quadratic equation may be resolved, by assuming for a and b, in the equations $(A,)$ and for a, b, and c, in the equations $(B,)$ such values as will render the general equation identical with the particular one under consideration, and substituting the values thus assumed in the final equation.

Simple quadratic equations are evidently solvible by the same *formulæ*, if we assume the co-efficient in the second term $=0$.

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* The value of x, in the third case, is ambiguous; because, $\sqrt{x^2-2ax+a^2}=x-a$, or $a-x$, and $\sqrt{4a^2x^2-4abx+b^2}=2ax-b$, or $b-2ax$; and there is nothing in the general equation, to show whether x is greater or less than a in the former case, or than $\dfrac{b}{2a}$ in the latter. The ambiguity does not exist in the second case, because $\sqrt{a^2+b}$ must be greater than a, and $\sqrt{b^2+4ac}$ greater than b, and therefore, cannot be equal respectively to $a-x$, and $b-2ax$, unless x be negative. When the sign of x is doubtful, the first and second cases are also ambiguous.

17. Given $9x^2-7x=116$.

Dividing by 9, $x^2-\frac{7}{9}x=\frac{116}{9}$

Comparing this equation with the 2d form, (A,) we have

$$2a=\frac{7}{9}, \text{ and } b=\frac{116}{9}$$

Hence, $x=\frac{7}{18}+\sqrt{(\frac{49}{324}+\frac{116}{9})}=\frac{72}{18}=4$.

Or comparing the given equation with the 2d form (B.)

$$a=9,\ b=7,\ c=116.$$

Hence, $x=\frac{7+\sqrt{49+4176}}{18}=\frac{7+65}{18}=4$.

18. Given $\sqrt{10+x}-\sqrt[4]{10+x}=2$, to find the value of x.

Considering $\sqrt[4]{10+x}$ as the quantity sought, and comparing the equation with the 2d form, (A,) we have

$$2a=1,\ b=2.$$

$$\therefore \sqrt[4]{10+x}=\frac{1}{2}+\sqrt{2\tfrac{1}{4}}=2.$$

$$10+x=2^4=16, \text{ and therefore, } x=6.$$

Or $\sqrt{10+x}-2=\sqrt[4]{10+x}$.

Hence by involution, $10+x-4\sqrt{10+x}+4=\sqrt{10+x}$.

Hence, $14+x=5\sqrt{10+x}$.

By involution, $196+28x+x^2=250+25x$.

Hence, $x^2+3x=54$.

Comparing this with the 1st form, (A,)

$$2a=3, \text{ and } b=54.$$

$$\therefore x=-\frac{3}{2}+\sqrt{(54\tfrac{9}{4})}=\frac{12}{2}=6.$$

19. Given $x+\sqrt{5x+10}=8$, to determine the value of x.

Multiplying by 5, and adding 10 to each member,

$$5x+10+5\sqrt{5x+10}=50.$$

Considering $\sqrt{5x+10}$ as the quantity sought; and comparing the equation with the 1st form, (A,) we have

$$2a=5, \text{ and } b=50.$$

Hence, $\sqrt{5x+10}=-\frac{5}{2}+\sqrt{(50\frac{25}{4})}=5.$

By involution and transposition, $5x=25-10.$

Therefore, $x=3.$

Otherwise.—By transposition, $\sqrt{5x+10}=8-x.$

By involution, $5x+10=64-16x+x^2,$

Or $x^2-21x=-54.$

Comparing this with the 2d form, (B,)

$$a=1, b=21, \text{ and } c=54.$$

Hence, $x=\frac{21\pm\sqrt{441-216}}{2}=18$, or 3.*

20. Given $\begin{cases} x^2y^2+xy=1260, \\ x+y=12, \text{ to find } x \text{ and } y. \end{cases}$

Comparing the 1st equation with form 1. (A,)

$$a=\tfrac{1}{2}, b=1260.$$

Hence, $xy=-\frac{1}{2}+\sqrt{(\frac{1}{4}+1260)}=35.$

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* The ambiguity is *introduced* into this solution by the involution of the given equation; for $(8-x)=(x-8)^2$: and one of the values of x, corresponds to the supposition, that $5x+10=(x-8)^2$ and the other to the true one, that $5x+10=(8-x)^2$.

Subtracting 4 times this equation from the square of the second, we have

$$x^2-2xy+y^2=4.$$

$$\therefore x-y=\pm 2, \text{ and } x=7 \text{ or } 5.$$

21. Given $\left\{\begin{matrix}\frac{3x^2}{y^2}-\frac{5x}{y}=2\\ xy=8.\end{matrix}\right\}$ to find the values of x and y.

Comparing the 1st equation with form 2d, $(B,)$

$$a=3,\ b=5,\ c=2.$$

Hence, $\frac{x}{y}=\frac{5+\sqrt{25+24}}{6}=2.$

$$\therefore x=2y, \text{ and } 2y^2=8.$$

$$\therefore y=\sqrt{\frac{8}{2}}=2, \text{ and } x=4.$$

22. Given $\left\{\begin{matrix}x^2+xy=12,\\ xy-2y^2=1,\end{matrix}\right\}$ to find x and y.

Assume $vy=x$, then $v^2y^2=x^2$, and $vy^2=xy$.

$$\therefore v^2y^2+vy^2=12, \text{ or } y^2=\frac{12}{v^2+v}$$

$$vy^2-2y^2=1, \text{ or } y^2=\frac{1}{v-2}$$

Hence $\frac{12}{v^2+v}=\frac{1}{v-2}$, or $12v-24=v^2+10$.

Therefore, $v^2-11v=-24$.

Comparing this with form 1st, $(A,)$ $a=-\frac{11}{2}$, $b=24$.

Whence, $v=\frac{11}{2}\pm\sqrt{(\frac{121}{4}-24)}=8$ or 3.

$$y=\sqrt{\frac{1}{v-2}}=\frac{1}{\sqrt{6}} \text{ or } 1,\ x=vy=\frac{8}{\sqrt{6}} \text{ or } 3.$$

23. Given $\left\{\begin{array}{l}\frac{x^2}{y}+\frac{y^2}{x}=a,\\ x+y=2b,\end{array}\right\}$ to find x and y.

Assume $x-y=2v$, then $x=b+v$; $y=b-v$.

Therefore, $\frac{(b+v)^2}{b-v}+\frac{(b-v)^2}{b+v}=a$.

And by multiplication, $(b+v)^3+(b-v)^3=a\times\overline{b-v}\times\overline{b+v}$.

Or $2b^3+6bv^2=ab^2-av^2$.

Whence, $v=\sqrt{\frac{ab^2-2b^3}{a+6b}}$

$\therefore\ x=b+\sqrt{\frac{ab^2-2b^3}{a+6b}}$, and $y=b-\sqrt{\frac{ab^2-2b^3}{a+6b}}$

24. Given $\left\{\begin{array}{l}\frac{x^2}{y}+x+y+\frac{y^2}{x}=a,\\ \frac{x^4}{y^2}+x^2+y^2+\frac{y^4}{x^2}=b,\end{array}\right\}$ to find x and y.

Assume $x+y=s$, $xy=p$.

Then $\frac{x^2}{y}+\frac{y^2}{x}=a-s$.

And multiplying by $xy=p$, $x^3+y^3=ap-sp$.

But, $x^3+y^3=s^3-3sp$.*

$\therefore\ ap-sp=s^3-3sp$, and $p=\frac{s^3}{a+2s}$ (C.)

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* $x+y=s$, $xy=p$.

From the square of the first equation subtract twice the second,

$$x^2+y^2=s^2-2p.$$

From this equation subtract the second, and

$$x^2-xy+y^2=s^2-3p.$$

Multiply by the first, and $x^3+y^3=s^3-3sp$.

Again, $x^2+y^2=s^2-2p$.

Also, $\frac{x^4}{y^2}+\frac{y^4}{x^2}=(a-s)^2-2p=a^2-2as+s^2-2p$.

Hence $\frac{x^4}{y^2}+x^2+y^2+\frac{y^4}{x^2}=a^2-2as+2s^2-4p=b$.

By substitution, (C_1) $a^2-2as+2s^2-\frac{4s^3}{a+2s}=b$.

Clearing the equation of fractions, $a^2-2as^2=ab+2bs$.

$$\therefore s^2+\frac{b}{a}s=\frac{a^2-b}{2}$$

Consequently, $s=\frac{-b}{2a}+\sqrt{(\frac{b^2}{4a^2}+\frac{a^2-b}{2})}$

Whence, $p=\frac{s^3}{a+2s}$ becomes known.

Now, $(x-y)^2=(x+y)^2-4xy=s^2-4p$.

Or, $x-y=\pm\sqrt{s^2-4p}$.

$$\therefore x=\frac{s\pm\sqrt{s^2-4p}}{2}, \text{and } y=\frac{s\mp\sqrt{s^2-4p}}{2}$$

Questions producing quadratic equations.

1. Required to find two numbers, such that their sum added to their product shall be 47; and the sum of the squares shall be 74?

Let x and y denote the numbers.

Then $x+y+xy=47$,
And $x^2+y^2=74$, } by the data.

Add twice the first equation to the second, and

$$x^2+2xy+y^2+2x+2y=168.$$

Or, $(x+y)^2+2.(x+y)=168$.

Hence form 1, (A_1) $x+y=-1+13=12$.

This equation subtracted from the first, leaves

$$xy=35.$$

By subtracting twice the last equation from the 2d,

$$x^2-2xy+y^2=4.$$

Hence by evolution, $x-y=\pm 2$.

$$\therefore x=7 \text{ or } 5.$$

2. A. and B. sold 130 ells of silk, for 42 crowns; 40 ells belonged to A. and 90 to B. Now, A. sold for a crown $\frac{1}{3}$ of an ell more than B. did. How many ells did each sell for a crown?

Let $x=$ the number B. sold,
Then $x+\frac{1}{3}=$the number A. sold, } for a crown.

$$\therefore \frac{90}{x}=\text{number of crowns received for B's. silk.}$$

$$\text{And } \frac{40}{x+\frac{1}{3}}=\frac{120}{3x+1}=\text{number received for A's. silk.}$$

$$\therefore \frac{90}{x}+\frac{120}{3x+1}=42,$$

$$270x+90+120x=126x^2+42x.$$

By transposition and division, $21x^2-58x=15$.

Comparing this with form 2, (B) we find

$$a=21,\ b=58,\ c=15.$$

Hence, $x=\frac{\sqrt{1260+3364}+58}{42}=\frac{126}{42}=3$, =number of ells B. sold for a crown.

$$\therefore 3\tfrac{1}{3}, \text{ number A. sold for a crown.}$$

3. There are two square yards, paved with stone, each stone being a foot square; the side of one yard exceeds that of the other by 12 feet, and the number of stones in the two is 2120; what are the sides?

Ans. 26 and 38 feet.

4. A labourer dug two trenches, one of which was 6 yards longer than the other, for £17 16*s.*, and the digging of each cost as many shillings per yard, as there were yards in its length; what was the length of each?

Ans. 10 and 16 yards.

5. A. and B. set out from two towns, which were distant from each other 247 miles, and travelled till they met. A. travelled 9 miles a day, and the number of miles walked by B. in a day, increased by 3, was equal to the number of days occupied by the journey. Required the number of days they were travelling, and the number of miles passed over by each?

Ans. 13 days, and A. went 117 miles, and B. 130.

6. A. and B. having bought 41 oxen, for which each of them paid 420 dollars, find A's. worth a dollar a head more than B's.; how must they divide them?

Ans. A. 20, B. 21.

7. Divide the number 14 into two parts, whose product shall be 48. Ans. the numbers are 6 and 8.

8. Given the sum of two numbers =9, and the sum of the squares =45; what are the numbers? Ans. 6 and 3.

9. What two numbers are those, whose sum, product, and the difference of their squares are equal to each other? Ans. $\frac{3}{2}+\frac{1}{2}\sqrt{5}$, and $\frac{1}{2}+\frac{1}{2}\sqrt{5}$.

9. There are four numbers in arithmetical progression, of which the product of the two extremes is 22, and that of the means 40; what are the numbers?

Ans. 2, 5, 8, and 11.

10. There are three numbers in geometrical progression, whose sum is 7, and the sum of their squares 21; what are the numbers? Ans. 1, 2, and 4.

11. Required to find two numbers, such that the less may be to the greater, as the greater is to 12; and the sum of the squares may be 45. Ans. 3 and 6.

12. What two numbers are those, whose difference is 2, and the difference of their cubes 98? Ans. 3 and 5.

13. What two numbers are those, whose sum is 6, and the sum of whose cubes is 72? Ans. 2 and 4.

14. Required to find two such numbers, that their product shall be 20, and the difference of their cubes 61?

Ans. 4 and 5.

15. Required to divide the number 5 into two such parts, that if each part be divided by the other, the sum of the quotients shall be $2\frac{1}{6}$? Ans. 3 and 2.

16. Divide 12 into two parts, so that their product may be equal to 8 times their difference. Ans. 8 and 4.

17. There are two numbers, the sum of whose squares is 89, and their sum multiplied by the greater, is 104; what are the numbers?

Ans. $\frac{1}{2}\sqrt[3]{2}$, and $\frac{3}{2}\sqrt[3]{2}$, or 8 and 5.

18. What number is that, which being divided by the product of its two digits, the quotient is $5\frac{1}{3}$; but when 9 is subtracted from it, the remainder is expressed by the same digits in an inverted order? Ans. 32.

19. Required to divide 18 into three such parts, that their squares may be equidifferent, and the sum of those squares may be 75. Ans. 1, 5, 7.

20. The sum of three equidifferent numbers is 12, and the sum of their 4th powers 962; what are the numbers?

Ans. 3, 4, and 5.

21. There are three equidifferent numbers, such that the square of the least, added to the product of the other two makes 28; but the square of the greatest, added to the product of the other two makes 44; what are the numbers? Ans. 2, 4, 6.

22. Three merchants, A. B. C., on comparing their gains, find they amount collectively, to 1,444 dollars; that B's. gain added to the square root of A's. makes 920 dollars; and that B's. gain added to the square root of C's. makes 912 dollars; how much did they severally gain? Ans. A. 400, B. 900, C. 144.

23. What two numbers are those, whose sum added

to the square of the sum, makes 702, and whose difference subtracted from the square of the difference, leaves 56?

Ans. 17 and 9.

24. The sum of two numbers is 10, and the sum of their 4th powers 1552; what are the numbers?*

Ans. 6 and 4.

25. There are four numbers in arithmetical progression, the common difference of which is 4, and their continued product 9945; what are the numbers? Ans. 5, 9, 13, 17.

26. The sum of three numbers in harmonical proportion is 191, and the product of the first and last is 4032; what are the numbers? Ans. 56, 63, 72.

27. The sum of two numbers added to their product makes 31; but the sum subtracted from the sum of the squares, leaves 48. Quere the numbers?

Ans. 7 and 3.

28. Required to find three numbers in continued proportion, whose sum shall be 26, and the sum of their squares 364? Ans. 2, 6, and 18.

29. Required to find two numbers, whose product shall be 320, and the difference of their cubes to the cube of their difference, as 61 to 1? Ans. 20 and 16.

30. If 700 dollars be divided among four persons, so that their shares may be in geometrical progression, and the difference of the extremes to the difference of the means as 37 to 12; what will be the several shares?

Ans. 108, 144, 192, and 256 dollars.

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* In examples of this nature, we may assume letters to denote the half sum, and half difference of the required numbers; whence, expressions for the numbers themselves, are readily obtained, and these being involved, and the powers added together, the odd powers of the unknown quantity will disappear. Hence, if the unknown quantity does not rise higher than the 5th power, the solution may be effected by quadratic equations.

31. The sum of two numbers is 11, and the sum of their 5th powers 17831; quere the numbers?

Ans. 7 and 4.

32. The difference of two numbers is 8, and the difference of their 4th powers is 14560; required the numbers?* Ans. 11 and 3.

33. What number is that, from the square of which, if 9 be subtracted, and the remainder be multiplied by the number itself, the product will be 80? Ans. 5.

34. Required a number, from the square of which, 30 being deducted, and the remainder multiplied by the number itself, the product shall be 56? Ans. $2+3\sqrt{2}$.

35. The continued product of five equidifferent numbers is 945, and their sum 25; what are the numbers?

Ans. 1, 3, 5, 7, and 9.

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* In this example, assuming for the half sum and half difference, we obtain a cubic equation, whence it appears that questions of this nature are not generally solvible by quadratics. When, however, we have an equation of the form $x^3+ax=b$; if $b=mn$, and $m^2+a=n$, the equation may be reduced to a quadratic. For multiplying by x, and adding m^2x^2 to each member, the given equation becomes $x^4+ax^2+m^2x^2=m^2x^2+bx$, or $x^4+nx^2=m^2x^2+nmx$, whence, by completing the squares and extracting the roots, $x^2+\frac{n}{2}=mx+\frac{n}{2}$, or $x=m$.

In the above example, x being made = half the sum of the required numbers, we find $x^3+16x=455=7\times65$; where $7^3+16=65$, whence, $x=7$. If $x^3-ax=mn$, and $m^2-a=n$; or if $x^3-ax=-mn$, and $a-m^2=n$, the equation may be reduced to a quadratic, and x found $=m$, as before. But if $x^3-ax=mn$, and $a-m^2=n$, or $x^3-ax=-mn$, and $m^2-a=n$; the equation may be changed to a quadratic, from which a second quadratic $x^2-mx=\pm n$, will arise, whence $x=\frac{m\pm\sqrt{m^2\pm4n}}{2}$.

36. The sum of three numbers in geometrical progression is 35, and the mean is to the difference of the extremes as 2 to 3; what are the numbers?

Ans. 5, 10, and 20.

37. The sum of three numbers in geometrical progression is 13, and the product of the mean by the sum of the extremes is 30; required the numbers?

Result, 1, 3, 9.

38. There is a number consisting of three digits in geometrical progression; the number itself, is to the sum of its digits, as 124 to 7; and if 594 be added to it, the order of the digits will be inverted; what is the number?

Ans. 248.

Promiscuous examples to exercise the foregoing rules.

1. What number is that, to the double of which, if 44 be added, the sum will be equal to 4 times the number proposed? Ans. 22.

2. A gentleman meeting 4 poor persons, divided a dollar among them in such manner, that their several shares composed an equidifferent series; and the sum given to the last was 4 times that given to the first. Quere their several shares?

Ans. 10, 20, 30, and 40 cents.

3. A sum of money being divided among 6 poor persons, the second received 10*d.*, the third 14*d.*, the fourth 25*d.*, the fifth 28*d.* and the sixth 33*d.* less than the first. The whole sum divided, was 10*d.* more than three times what the first received. How much did they severally receive? Ans. 40*d.*, 30*d.*, 26*d.*, 15*d.*, 12*d.*, and 7*d.*

4. A mercer having cut 19 yards from each of 3 equal pieces of silk, and 17 yards from another of the same length, finds the four remnants together, measure 142 yards; what was the original length of each piece?

Ans. 54.

5. A grazier having two flocks of sheep, containing the same number; sells 39 from one, and 93 from the

other, and then finds one flock twice as numerous as the other; what number did each of them contain at first?

Ans. 147.

6. From each of 16 coins, an artist filed the value of 20 cents; when the coins, being examined, were found worth only 11 dollars 68 cents; what was the original value of each? Ans. 93 cents.

7. The hold of a ship, containing 442 gallons, is emptied in 12 minutes by two buckets; the greater of which, holding twice as much as the less, is emptied twice in 3 minutes, and the less, is emptied three times in 2 minutes. Quere the number of gallons held by each bucket? Ans. 26 and 13.

8. A trader maintained himself for 3 years at an annual expense of £50; and in each of those years, augmented that part of his stock which was not expended by $\frac{1}{3}$ thereof. At the end of the third year, his original stock was doubled; what was that stock?

Ans. £740.

9. What number is that, which being added to, and subtracted from 36, the sum of the cube roots of the results shall be 6? Ans. 28.

10. Required to find a fraction, to the numerator of which if 4 be added, the value will be $\frac{1}{2}$; but if 7 be added to the denominator, the value will be $\frac{1}{3}$? Ans. $\frac{5}{18}$.

11. What two numbers are those, whose difference, sum, and product, are as the numbers 2, 3, and 5 respectively? Ans. 10 and 2.

12. A vintner having mixed a quantity of brandy and water, finds that if he had mixed 6 gallons more of each, he would have had 7 gallons of brandy for every 6 of water; but if he had mixed 6 gallons less of each, he would have had 6 gallons of brandy for every 5 of water. Quere the number of gallons of each?

Ans. 78 of brandy, and 66 of water.

13. A. and B. together, are able to perform a piece of work in 15 days; after working jointly 6 days, B. finishes

it alone in 30 days; in what time would each of them singly effect it? Ans. A. in $21\frac{7}{9}$ days, B. in 50 days.

14. A company of smugglers found a cave which would exactly hold their cargo, viz. 13 bales of cotton, and 33 casks of rum; but while they were unloading, a revenue cutter appeared, on which they sailed away with 9 casks and 5 bales, having filled $\frac{2}{3}$ of the cave; how many bales, or how many casks would the cave contain?

Ans. 24 bales, or 72 casks.

15. There are two numbers, whose sum is to their difference as 8 to 1; and the difference of whose squares is 128; what are the numbers? Ans. 18 and 14.

16. Required those two numbers, whose sum is to the less as 5 to 2; and whose difference multiplied by the difference of their squares is 135? Ans. 9 and 6.

17. A merchant laid out a certain sum upon a speculation, and found, at the end of a year, that he had gained £69. This he added to his stock, and at the end of the second year, found he had gained as much per cent. as in the first. Continuing in this manner, and each year adding to his stock the gain of the preceding, he found at the end of the fourth year, that his stock was to the sum first laid out, as 81 to 16. Quere the sum first invested? Ans. £138.

18. There are two numbers, whose sum is to the greater as 40 is to the less, and whose sum is to the less as 90 to the greater; what are the numbers?

Ans. 36 and 24.

19. The area of a rectangular parallelogram is 960 yards, and the length exceeds the breadth by 16 yards; what are the sides? Ans. 40 and 24.

20. The area of a rectangular parallelogram is 480, and the sum of the length and breadth 52. Quere the sides? Ans. 40 and 12.

21. The sides of a right angled triangle, form an equidifferent series, whose common difference is 3; what are the sides? Ans. 15, 12 and 9.

22. There are three numbers, the difference of whose differences is 5, their sum is 20, and their continued product 130; required the numbers?

Ans. 2, 5, and 13.

23. The sum of three numbers is 21, the sum of the squares of the greatest and least is 137; and the difference of the the differences is 3. Quere the numbers?

Ans. 4, 6, and 11.

24. There is a number consisting of two digits, which divided by the sum of its digits, has a quotient greater by 2, than the first digit. But, the digits being inverted and divided by the sum of the digits increased by unity, the quotient is equal to the first digit increased by 4. Quere the number? Ans. 24.

25. Required to find four numbers in geometrical progression, whose sum shall be 15, and the sum of the squares 85? Ans. 1, 2, 4, and 8.

26. There are five numbers in geometrical progression, whose sum is 242, and the sum of their squares 29524; what are the numbers? Ans. 2, 6, 18, 54, 162.

27. There are six numbers in geometrical progression, the sum of the extremes is 99, and the sum of the other four terms is 90. Quere the numbers?

Ans. 3, 6, 12, 24, 48, and 96.

28. The American dollar consists of 1485 parts by weight, of pure silver, and 179 of copper. Quere the specific gravity of this dollar, the specific gravity of pure silver being 11092, and that of copper 9000?

Ans. 10821.

Section VIII.

ON RATIOS.

61. Ratio is the relation that two quantities of the same kind bear to each other.

Ratio is estimated by the quotient arising from the division of the first term by the second. Thus, if $\frac{a}{b}=\frac{c}{d}$, a is said to have the same ratio to b, that c has to d; or the quantities a, b, c, d, are termed *proportionals;* which is briefly expressed thus, $a:b::c:d$.

The first and third terms are called *antecedents*, the second and fourth *consequents.**

62. When four quantities are proportionals, the product of the first and fourth is equal to the product of the second and third; and reciprocally,

If $\frac{a}{b}=\frac{c}{d}$, by multiplying by bd, $ad=bc$.

Reciprocally, if $ad=bc$, $\frac{ad}{bd}=\frac{bc}{bd}$, or $\frac{a}{b}=\frac{c}{a}$

63. When four quantities are proportionals, the sum of the first and second is to the second, as the sum of the third and fourth is to the fourth.

$$\text{If } \frac{a}{b}=\frac{c}{d};\ \frac{a}{b}+1=\frac{c}{d}+1, \text{ or } \frac{a+b}{b}=\frac{c+d}{d}$$

$$\text{Or, } a+b:b::c+d:d.$$

If the word difference be substituted for sum, the proposition will still be a true one.

$$\text{For } \frac{a}{b}=\frac{c}{d}, \therefore\ \frac{a}{b}-1=\frac{c}{d}-1, \text{ or } \frac{a-b}{b}=\frac{c-d}{d},$$

$$\text{Or, } a-b:b::c-d:d.$$

64. If four quantities be proportionals, the sum of the first and second is to their difference, as the sum of the third and fourth is to their difference.

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* The terms are similarly designated when more than four are concerned.

For by the last article, $\frac{a+b}{b}=\frac{c+d}{d}$, and $\frac{a-b}{b}=\frac{c-d}{d}$

And dividing the former by the latter,

$$\frac{a+b}{a-b}=\frac{c+d}{c-d}, \text{ or } a+b : a-b :: c+d : c-d.$$

65. When any number of quantities are proportionals, as one antecedent is to its consequent, so is the sum of all the antecedents to the sum of all the consequents.

Let $a : b :: c : d :: e : f :: g : h$, &c., then (art. 62.)

$$ad=bc,\ af=be,\ ah=bg, \text{ &c., also } ab=ba.$$

$$\therefore\ ab+ad+af+ah, \text{ &c. } =ba+bc+be+bg, \text{ &c.}$$

Or, $a\times\{b+d+f+h, \text{ &c.}\}=b\times\{a+c+e+g, \text{ &c.}\}$

$\therefore$ (art. 62,) $a : b :: a+c+e+g, \text{ &c.} : b+d+f+h, \text{ &c.}$

66. When four quantities are proportionals, if the first and second be multiplied or divided by any quantity, and likewise the third and fourth, the resulting quantities will be proportionals.

If $a : b :: c : d, \frac{a}{b}=\frac{c}{d}$, but $\frac{a}{b}=\frac{ma}{mb}$;

And $\frac{c}{d}=\frac{nc}{nd}$, $\therefore \frac{ma}{mb}=\frac{nc}{nd}$, or $ma : mb :: ne : nd.$

The demonstration is manifestly applicable when m and n are, one or both, fractional numbers.

67. When the first and third of four proportionals, are multiplied or divided by any number, and also, the second and fourth, the resulting quantities are proportionals.

For $\frac{a}{b}=\frac{c}{d}$ $\therefore \frac{ma}{b}=\frac{mc}{d}$ and $\frac{ma}{nb}=\frac{mc}{nd}$

Or, $ma : nb :: mc : nd$, m and n being any numbers, either integral or fractional.

68. If four quantities be proportionals, the like powers or roots of these quantities will be proportionals.

If $a:b::c:d$, then $\frac{a}{b}=\frac{c}{d}$, and $\frac{a^n}{b^n}=\frac{c^n}{d^n}$

Also, $\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}=\frac{c^{\frac{1}{n}}}{d^{\frac{1}{n}}}$, or $a^n : b^n :: c^n : d^n$.

And $a^{\frac{1}{n}} : b^{\frac{1}{n}} :: c^{\frac{1}{n}} : d^{\frac{1}{n}}$.

69. A ratio compounded of several ratios is indicated by the continued product of the quantities which denote the component ratios. Thus, the ratio compounded of the ratios of $a : b$, of $c : d$, and of $e : f$, is indicated by

$$\frac{a}{b}\times\frac{c}{d}\times\frac{e}{f}, \text{ or } \frac{ace}{bdf}$$

The ratio which the first of a series of quantities, of like kind, has to the last, is the same as the ratio compounded of the ratios of the first to the second, of the second to the third, &c. to the last.

Let a, b, c, d, e, f, be quantities of a like kind, then

$$\frac{a}{f}=\frac{a}{b}\times\frac{b}{c}\times\frac{c}{d}\times\frac{d}{e}\times\frac{e}{f}$$

70. In two ranks of proportionals, if the corresponding terms be multiplied together, the products will be proportionals.

Let $a : b :: c : d$, $e : f :: g : h$.

Then $\frac{a}{b}=\frac{c}{d}$, $\frac{e}{f}=\frac{g}{h}$ $\therefore \frac{ae}{bf}=\frac{cg}{dh}$

Or, $ae : bf :: cg : dh$.

The demonstration may be easily applied to any number of proportions.

71. A ratio compounded of two, three, four, &c. equal ratios, is called the duplicate, triplicate, quadruplicate, &c. of one of the component ratios.

The ratio compounded of any number of equal ratios is the same as the ratio of such power of the first term, as is indicated by the number of component ratios, to a like power of the second.

Let $a : b :: b : c$, then $\frac{a}{b}=\frac{b}{c}$

$\therefore$ (art. 67,) $\frac{a^2}{b^2}=\frac{ab}{bc}=\frac{a}{c}$, or $a^2 : b^2 :: a : c$.

Again, let $a : b :: b : c :: c : d :: d : e$, &c.

Then, $\frac{a}{b}=\frac{a}{b}$, $\frac{a}{b}=\frac{b}{c}$, $\frac{a}{b}=\frac{c}{d}$, $\frac{a}{b}=\frac{d}{e}$, $\frac{a}{b}=\frac{e}{f}$, &c. to n equations; and multiplying the first and second members respectively together,

$$\frac{a^n}{b^n}=\frac{a}{b}\times\frac{b}{c}\times\frac{c}{d}\times\frac{d}{e}\times\frac{e}{f}=\frac{a}{f}$$

Or, $a^n : b^n :: a : f$.

72. When the ratio of the first of three quantities to the second, is the same as the ratio of the second to the third, the ratio of the first to the second is termed the *sub-duplicate* of the ratio of the first to the third.

When four quantities are continued proportionals, the ratio of the first to the second is called the *sub-triplicate* of the ratio of the first to the fourth.

A ratio compounded of a simple and sub-duplicate ratio, is called a *sesquiplicate ratio.*

73. The sub-duplicate ratio is equivalent to the ratio of the square roots; the sub-triplicate, to the ratio of the cube roots; and the sesquiplicate, to the ratio of the square roots of the cubes.

Let $a : b :: b : c :: c : d$. Also, $b : e :: e : c$.

First, (art. 71,) $\frac{a^2}{b^2}=\frac{a}{c}$, $\therefore \frac{a}{b}=\frac{a^{\frac{1}{2}}}{c^{\frac{1}{2}}}$, or $a : b :: a^{\frac{1}{2}} : c^{\frac{1}{2}}$.

In like manner, $\frac{a}{b}=\frac{a^{\frac{1}{3}}}{d^{\frac{1}{3}}}$, or $a:b::a^{\frac{1}{3}}:d^{\frac{1}{3}}$.

Also, since $b:e::e:c$, and $a:b::c:d$.

$e^2=bc=ad$, (art. 62.) But $\frac{a^2}{b^2}=\frac{a}{d}=\frac{a^2}{ad}=\frac{a^2}{e^2}$

$\therefore \frac{a^{\frac{3}{2}}}{b^{\frac{3}{2}}}=\frac{a}{e}$, or $a^{\frac{3}{2}}:b^{\frac{3}{2}}::a:e$, in which the ratio of $a:e$, is compounded of the ratios of $a:b$, and of $b:e$, or of the ratio of $a:b$, and the sub-duplicate of the ratio of $b:c$.

ON THE VARIATIONS OF QUANTITIES.

74. In the investigation of the relation which varying and dependent quantities bear to each other, the conclusions are more readily obtained, by expressing only two terms in each proportion, than by retaining the four. But though, in considering the variation of such quantities, two terms only are expressed, it must be remembered, that four are supposed; and that the operations, by which our conclusions are obtained, are in reality the operations of proportionals.

75. One quantity is said to vary *directly*, as another, when one is such a function* of the other, that if the former be changed, the latter will be changed in the same ratio. Thus, if B be such a function of A, that by changing A to a, B shall be changed to b; making $A:a::B:b$,

* The *function* of any variable quantity x, is an algebraic expression, in which x, combined with invariable quantities, is involved. Thus, $1+x$, $(1+x)^n$, ax, x^n, a^x, &c. are functions of x. Analysts sometimes use the Greek letter ϕ, to denote a function. Thus, ϕx may represent any function of x.

A is said to vary *directly* as B. This relation is designated thus, $A \propto B$.

76. One quantity is said to vary *inversely* as another, when the latter is such a function of the former, that the one being increased or diminished, the other will be diminished or increased in the same ratio. Thus, if B be such a function of A, that by changing A to a, B becomes changed to b; making $A : a :: b : B$. Then A is said to vary inversely as B. Indicated thus, $A \propto \frac{1}{B}$

77. One quantity is said to vary as two others jointly, when the two last are such functions of the first, that, the ratio which any two values of the first bear to each other, shall be the same as the ratio compounded of the ratios of the corresponding values of the other two. Thus, A varies as B and C jointly, ($A \propto BC$,) when A being changed to a, B changes to b, and C to c, so that

$$\frac{A}{a} = \frac{B}{b} \times \frac{C}{c}$$

78. One quantity is said to vary directly as a second, and inversely as a third, when the ratio which any two values of the first bear to each other, is the same as the ratio compounded of the direct ratio of the corresponding values of the second, and the inverse ratio of those of the third.

Thus, A varies directly as B, and inversely as C, ($A \propto \frac{B}{C}$,) when A, B, C; a, b, c, being corresponding values,

$$\frac{A}{a} = \frac{B}{b} \times \frac{c}{C}$$

In the following articles, A, B, C, &c. represent corresponding values of any quantities, and a, b, c, &c. any other corresponding values of the same quantities.

79. If $A \propto B$, and m, n denote any given numbers,

Then $A \propto mB \propto \frac{B}{n}$

For since $\frac{A}{a}=\frac{B}{b}$, and $\frac{B}{b}=\frac{mB}{mb}=\frac{\frac{1}{n}B}{\frac{1}{n}b}$; $\frac{A}{a}=\frac{mB}{mb}=\frac{\frac{1}{n}B}{\frac{1}{n}b}$

80. If $A\propto B$, then $A^n\propto B^n$; and $A^{\frac{1}{n}}\propto B^{\frac{1}{n}}$. See art. 68.

81. If $A\propto B$, and $C\propto D$, then $AC\propto BD$. See art. 70.

82. If $A\propto BC$, then $B\propto\frac{A}{C}$, and $C\propto\frac{A}{B}$

For $\frac{A}{a}=\frac{B}{b}\times\frac{C}{c}$. $\therefore\ \frac{Ac}{aC}=\frac{B}{b}$, or $\frac{B}{b}=\frac{A}{a}\times\frac{c}{C}$ $(B\propto\frac{A}{C})$

Also $\frac{Ab}{aB}=\frac{C}{c}$, or $\frac{C}{c}=\frac{A}{a}\times\frac{b}{B}$, $(C\propto\frac{A}{B})$

From the three preceding articles it appears, that quantities connected by the sign $\propto$, may be treated as the members of an equation, as far as multiplication, division, involution or evolution is concerned.

83. If $A\propto C$, and $B\propto C$, A and B being quantities of the same kind, then $(A\pm B)\propto C$, and $\surd AB\propto C$.

For since $\frac{A}{a}=\frac{C}{c}$, and $\frac{B}{b}=\frac{C}{c}$; $\frac{A}{a}=\frac{B}{b}$. $\therefore\ \frac{A}{B}=\frac{a}{b}$

And $\frac{A}{B}\pm1=\frac{a}{b}\pm1$, or $\frac{A\pm B}{B}=\frac{a\pm b}{b}$. $\therefore\ \frac{A\pm B}{a\pm b}=\frac{B}{b}=\frac{C}{c}$

Again, since $A\propto C$, and $B\propto C$, (art. 81,) $AB\propto C^2$.

$\therefore$ (art. 80,) $\surd AB\propto C$.

84. If while A and B vary $AB=$ a constant quantity, $A\propto\frac{1}{B}$, and $B\propto\frac{1}{A}$.

For since $AB=ab$, $\frac{A}{a}=\frac{b}{B}$, $(A\propto\frac{1}{B},)$

And $\frac{B}{b}=\frac{a}{A}$ $(B\propto\frac{1}{A}.)$

Section IX.

OF SERIES.

85. From the nature of powers, (art. 14,) we readily discover that any two powers of the same root, multiplied together, produce a power indicated by the sum of the exponents of the factors; thus, $a^3.a^3=a^5$; $a^m \times a^n = a^{m+n}$.

On the other hand, $\frac{a^5}{a^3}=\frac{1.a.a.a.a.a}{1.a.a.a}=aa=a^2$, $\frac{a^m}{a^n}=a^{m-n}$.

Suppose $n=m+v$, then $\frac{a^m}{a^n}=\frac{a^m}{a^m.a^v}=\frac{1}{a^v}$

But in this case $m-n=-v$, if then $\frac{a^m}{a^n}$ can be always denoted by a^{m-n}; a^{-v} must be equivalent to $\frac{1}{a^v}$

Considering the consecutive powers of a given root a as the terms of a geometrical series, extending at pleasure, above and below unity.

$$\frac{1}{a^3}, \frac{1}{a^2}, \frac{1}{a}, 1, a, a^2, a^3, \text{\&c.}$$

Or the equivalent series, $a^{-3}, a^{-2}, a^{-1}, a^0, a^1, a^2, a^3$, &c.

The index denotes the order of the term beginning with unity, and making the index positive or negative, according as the quantity is above or below the point of commencement.

86. Between any two of these terms, let $n-1$ mean proportionals be interposed, ex. gr. between a^0 $(=1,)$ and a, calling the first x, then (art. 71,) $1 : a :: 1^n : x^n :: 1 : x^n$.

$\therefore a=x^n$.

Our series then becomes, (x^0 being $=1=a^0$.)

$$x^{-2n} \ldots x^{-n-1}, x^{-n}, \ldots . x^{-2}, x^{-1}, x^0, x^1, x^2;$$

$$\ldots x^n, x^{n+1}, x^{n+2}, x^{2n}, x^{2n+1}, \&c.$$

Now, the product of any two terms of this series is, manifestly, indicated by x, with an exponent equal to the sum of the exponents of the factors, as $x^n \times x^{2n+1} = x^{3n+1}$;

$$x^{n+1} \times x^{-n} = \frac{x^{n+1}}{1} \times \frac{1}{x^n} = \frac{x^{n+1}}{x^n} = x^1 = x^{n+1-n}.$$

$$\frac{x^{2n+3}}{x^{n+2}} = x^{n+1} = x^{2n+3-(n+2.)}$$

$$\frac{x^{n+4}}{x^{-n}} = x^{n+4} \div \frac{1}{x^n} = x^{n+4} \times x^n = x^{2n+4} = x^{n+4+n}.$$

If for x and its powers we write $a^{\frac{1}{n}}$ and its powers, the series, though changed as to the form of its exponents, evidently retains the same essential character, and the operations of multiplication and division are performed by taking the sum and difference of the exponents of the factors, whether those exponents are positive or negative, integral or fractional. By thus arranging the powers as the terms of a geometrical series, we perceive that the exponent, whether integral or fractional, serves not only to designate the power, but to indicate the situation of the term, in relation to the unit's place.

87. From these principles we readily infer, that any letter or quantity may be removed from the denominator of a fraction to its numerator, and vice versa, by changing the sign of its exponent.

Thus, $\frac{a}{x^2} = \frac{a}{1} \times \frac{1}{x^2} = ax^{-2}$.

$$\frac{a+b}{(a^2-b^2)^{\frac{1}{2}}} = (a+b).(a^2-b^2)^{-\frac{1}{2}}. \quad ab^2 = \frac{a}{b^{-2}}$$

88. It frequently happens in the division or evolution of algebraic quantities, as well as in common arithmetic,

that to whatever extent the process may be continued, a remainder will still occur; in which case the resulting quotient or root, mostly assumes the form of an infinite series.

Thus, $\frac{1}{1-x}=1+x+x^2+x^3+x^4+$, &c. to infinity.

$$\sqrt{a^2+b^2}=a+\frac{b^2}{2a}-\frac{b^4}{8a^3}+\frac{b^6}{16a^5}-\frac{5b^8}{128a^7}+, \text{ &c.}^*$$

In some instances, when a few terms of the series are obtained, *the law of continuation*, or the relation of the successive terms to those which precede them, becomes manifest. The first of the series above given, may be readily continued to any proposed extent. The law of continuation in the last is not obvious on first view; it will, however, be shown further on.

89. In the investigations connected with series, the method of *indeterminate co-efficients* is often found particularly convenient. It depends upon the following theorem.

Let $Ax+Bx^2+Cx^3+Dx^4$, &c. $=ax+bx^2+cx^3+dx^4$, &c., the series being both infinite, or, if finite, extending to the same number of terms; and A, B, C, a, b, c, invariable; if then the above equation be true, whatever value may be assigned to x, A will be equal to a, $B=b$, &c.; for dividing by x, we have

$$A+Bx+Cx^2+Dx^3, \text{ &c.}=a+bx+cx^2+dx^3, \text{ &c.}$$

Now, the equation being true for all values of x, must hold if $x=0$, in which case, the equation becomes $A=a$; subtracting this equation from the given one,

$$Bx+Cx^2+Dx^3, \text{ &c.}=ax+bx^2+cx^3, \text{ &c.}$$

---

* When the successive terms of an infinite series continually decrease, it is called a converging series; in which case, the sum of the series may be approximated by collecting a finite number of terms.

Dividing by x, $B+Cx+Dx^2$, &c. $=b+cx+dx^2$, &c.

Whence, if $x=0$, $B=b$.

In the same manner $C=c$, $D=d$, &c.

This equation becomes by transposition,

$$\left.\begin{array}{l} Ax+Bx^2+Cx^3+Dx^4+, \text{ &c.} \\ -ax-bx^2-cx^3-dx^4+, \text{ &c.} \end{array}\right\} =0.$$

Hence it appears, that when all the terms of a general equation are brought to one side, the co-efficients of the several powers of the unknown quantity are respectively equal to 0.

This principle is applied in the following examples.

1. Required to express $\frac{1}{1-2x+x^2}$ in a series.

It is easy to perceive that the first term must be 1.

Assume then $\frac{1}{1-2x+x^2}=1+ax+bx^2+cx^3+dx^4$, &c.

Multiply by $1-2x+x^2$, and bring the terms to one side of the equation,

$$\left.\begin{array}{l} 1+ax+bx^2+cx^3+dx^4+, \text{ &c.} \\ \quad -2x-2ax^2-2bx^3-2cx^4-, \text{ &c.} \\ -1 \quad +x^2+ax^3+bx^4+, \text{ &c.} \end{array}\right\} =0.$$

Hence, $a-2=0, b-2a+1=0, c-2b+a=0, d-2c+b=0$.

$\therefore a=2, b=4-1=3, c=6-2=4, d=8-3=5$.

Whence, $\frac{1}{1-2x+x^2}=1+2x+3x^2+4x^3+5x^4+$, &c.

in which the *law of continuation* is manifest.

2. Required to develope $\sqrt{a^2+x^2}$, in a series.

$$\sqrt{a^2+x^2}=\sqrt{(a^2\times 1+\frac{x^2}{a^2})}=a\sqrt{(1+\frac{x^2}{a^2})}=a\sqrt{1+z^2},$$

Putting $\frac{x^2}{a^2}=z^2$.

Assume $\sqrt{1+z^2}=1+az^2+bz^4+cz^6+dz^8+$, &c.*

By involution and transposition,

$$\left.\begin{array}{r} 1+2az^2+2bz^4+2cz^6+2dz^8+2ez^{10}+, \text{ &c.} \\ a^2z^4+2abz^6+2acz^8+2adz^{10}, \text{ &c.} \\ b^2z^8+2bcz^{10}, \text{ &c.} \end{array}\right\}=0.$$

$$-1-z^2.$$

$$\therefore 2a-1=0, 2b+a^2=0, 2c+2ab=0, 2d+2ac+b^2=0, \text{ &c.}$$

Whence, $a=\frac{1}{2}$, $b=-\frac{1}{8}$, $c=\frac{1}{16}$, $d=-\frac{5}{128}$, $e=\frac{7}{256}$

$$\therefore \sqrt{1+z^2}=1+\frac{1}{2}z^2-\frac{1}{8}z^4+\frac{1}{16}z^6-\frac{5}{128}z^8+\frac{7}{256}z^{10}, \text{ &c.}$$

Consequently,

$$\sqrt{a^2+x^2}=a\left\{1+\frac{x^2}{2a^2}-\frac{x^4}{8a^4}+\frac{x^6}{16a^6}-\frac{5x^8}{128a^8}, \text{ &c.}\right\}=$$

$$a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}+\frac{7x^{10}}{256a^9}, \text{ &c.}$$

To find, if possible, the law of continuation, we observe that

$$-\frac{1}{8}=\frac{1}{2}\times\frac{-1}{4}, \quad \frac{1}{16}=\frac{1}{2}\times\frac{-1}{4}\times\frac{-3}{6},$$

$$-\frac{5}{128}=\frac{1}{2}\times\frac{-1}{4}\times\frac{-3}{6}\times\frac{-5}{8}, \quad \frac{7}{256}=\frac{1}{2}\times\frac{-1}{4}\times\frac{-3}{6}\times\frac{-5}{8}\times\frac{-7}{10}$$

---

* If the series $1+az+bz^2+cz^3$, &c. had been assumed, we should have found $a=0$, $c=0$, &c.

90. Required to express $\frac{x^n-y^n}{x-y}$ in a series, n being an integral number.

$$\frac{x^n-y^n}{x-y}=\frac{x^n}{x}\times\frac{1-\frac{y^n}{x^n}}{1-\frac{y}{x}}=x^{n-1}\times\frac{1-z^n}{1-z}$$

Assuming $z=\frac{y}{x}$

Now, if $\frac{1-z^n}{1-z}$ be resolved into a series, it is manifest the first term will be 1.

And therefore, $\frac{x^n-y^n}{x-y}=x^{n-1}\times\frac{1-z^n}{1-z}$ resolved into a series, will have its first term x^{n-1}.

Again, $\frac{x^n-y^n}{x-y}=\frac{y^n-x^n}{y-x}=\frac{y^n}{y}\times\frac{1-\frac{x^n}{y^n}}{1-\frac{x}{y}}=y^{n-1}\times\frac{1-v^n}{1-v}$,

making $v=\frac{x}{y}$, which resolved into a series, will have its first term y^{n-1}.

Whence it appears, that if $\frac{x^n-y^n}{x-y}$ be resolved into a series, the first and last terms will be x^{n-1}, and y^{n-1} repectively.

Assume then $\frac{1-z^n}{1-z}=1+az+bz^2+\ldots.pz^{n-2}+qz^{n-1}$.

Multiply by $(1-z,)$ and transpose $1-z^n$, whence,

$$\left.\begin{array}{llll} 1+az+bz^2+cz^3, & & pz^{n-2}+qz^{n-1}, & \\ -z-az^2-bz^3, & & -mz^{n-2}-pz^{n-1}-qz^n, & \\ -1 & & & +z^n, \end{array}\right\}=0.$$

$\therefore$ (art. 89,) $a-1=0$, $b-a=0$, $c-b=0$, $p-m=0$, $q-p=0$, $1-q=0$.

Or, $\frac{x^n-y^n}{x-y}=x^{n-1}\times\{1+z+z^2+z^3.\ \ z^{n-2}+z^{n-1}\}=$

$x^{n-1}+x^{n-2}y+x^{n-3}y^2+x^{n-4}y^3.\ \ x^{n-\overline{n-1}}y^{n-2}+y^{n-1}.$

Whence, $x^n-y^n=(x-y)\times\{x^{n-1}+x^{n-2}y+x^{n-3}y^2.$
$y^{n-1}.\}$

Here the indices of x descending regularly from $n-1$, to 0, or $n-n$, it appears the number of terms in the series is n.

91. Required to develope $(1+x)^n$ in a series, n being a whole positive number.

It is easy to perceive that the first term of the series is 1.

Assume then,

$(1+x)^n=1+ax+bx^2+cx^3+dx^4+$, &c. (*A.*)

Consequently, $(1+y)^n=1+ay+by^2+cy^3+dy^4$, &c.

By subtraction, $(1+x)^n-(1+y)^n=a(x-y)+b(x^2-y^2)$
$+c(x^3-y^3)+d(x^4-y^4,)$ &c.

Divide by $\overline{1+x}-\overline{1+y}=x-y$, and we shall have

$$\frac{(1+x)^n-(1+y)^n}{x-y}=a+b(\frac{x^2-y^2}{x-y})+$$

$$b(\frac{x^3-y^3}{x-y})+d(\frac{x^4-y^4}{x-y})\ \text{&c.}$$

$$\text{Or } (1+x)^{n-1}+(1+x)^{n-2}\times\overline{1+y}(1+y)^{n-1}$$
$$=a+b(x+y)+c(x^2+xy+y^2)$$
$$+d(x^3+x^2y+xy^2+y^3)+,\ \text{&c.}$$

This equation being true, whatever value may be assigned to x and y, let us suppose $x=y$, then this equation becomes

(art. 90,) $n.(1+x)^{n-1}=a+2bx+3cx^2+4dx^3+5ex^4$, &c.

Multiply by $(1+x)$

$$\therefore n.(1+x)^n=a+2bx+3cx^2+4dx^3+5ex^4, \text{ \&c.}$$
$$ax+2bx^2+3cx^3+4dx^4, \text{ \&c.}$$

But (eq. *A.*)

$$n.(1+x)^n=n+nax+nbx^2+ncx^3+ndx^4, \text{ \&c.}$$

Consequently, $a=n$, $b=a.\frac{n-1}{2}=n.\frac{n-1}{2}$,

$$c=b.\frac{n-2}{3}=n.\frac{n-1}{2}.\frac{n-2}{3};\ d=c.\frac{n-3}{4}=n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4};$$
$$e=d.\frac{n-4}{5}=n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}.\frac{n-4}{5}.$$

Hence, $(a+b)^n=a^n\times(1+\frac{b}{a})^n=$

$$a^n.\{1+n\frac{b}{a}+n.\frac{n-1}{2}\frac{b^2}{a^2}+, \text{ \&c.}\}=$$
$$a^n+na^{n-1}b+n.\frac{n-1}{2}a^{n-2}b^2+n.\frac{n-1}{2}.\frac{n-2}{3}a^{n-3}b^3+, \text{ \&c.}$$

When n is a whole positive number, as here supposed, the series is finite; the number of terms being $n+1$; for the co-efficient of term $\overline{n+2}$ is

$$n.\frac{n-1}{2}.\frac{n-2}{3}\ldots.\frac{n-n}{n+1}=0.$$

If $(a-b)^n$ be required, we may put $-b$ and its powers for $+b$ and its powers.

Whence, $(a-b)^n=a^n-na^{n-1}b+n.\frac{n-1}{2}a^{n-2}b^2-$

$$n.\frac{n-1}{2}.\frac{n-2}{3}a^{n-3}b^3+n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}a^{n-4}b^4, \text{ \&c.}$$

92. Required to develope $(1+x)^{\frac{m}{n}}$ in a series.

Assume $(1+x)^{\frac{m}{n}}=1+ax+bx^2+cx^3+dx^4+ex^5$, &c. (*A.*)

Then $(1+y)^{\frac{m}{n}}=1+ay+by^2+cy^3+dy^4+ey^5$, &c. (*B.*)

Put $(1+x)^{\frac{1}{n}}=v$, $(1+y)^{\frac{1}{n}}=w$, then $1+x=v^n$, $1+y=w^n$,

$$(1+x)^{\frac{m}{n}}=v^m,\ (1+y)^{\frac{m}{n}}=w^m.$$

Substituting these expressions in equations *A* and *B*, and subtracting, we have

$$v^m-w^m=a(x-y)+b(x^2-y^2)+c(x^3-y^3)+$$
$$d.(x^4-y^4)+, \text{ \&c.}$$

$$\therefore \text{(art. 90,) } (v-w)\times\{v^{m-1}+v^{m-2}w.\quad w^{m-1}\}=$$
$$(x-y).\{a+b(x+y)+c.(x^2+xy+y^2)+$$
$$d(x^3+x^2y+xy^2+y^3), \text{ \&c.}\}$$

But $x-y=\overline{1+x}-\overline{1+y}=v^n-w^n=(v-w)\times$
$$\{v^{n-1}+v^{n-2}w+.\quad w^{n-1}\}$$

$\therefore$ Dividing the last equation by this, we have

$$\frac{v^{m-1}+v^{m-2}w+v^{m-3}w^2.\quad w^{m-1}}{v^{n-1}+v^{n-2}w+v^{n-3}w^2.\quad w^{n-1}}=a+b(x+y)+$$
$$c(x^2+xy+y^2)+d(x^3+x^2y+xy^2+y^3), \text{ \&c.}$$

Now, suppose $x=y$, whence $v=w$, and our equation becomes (art. 90,)

$$\frac{mv^{m-1}}{nv^{n-1}}=\left(\frac{m}{n}.\frac{v^m}{v^n}\right)=a+2bx+3cx^2+4dx^3+5ex^4, \text{ \&c.}$$

Multiply by $v^n = 1+x$, and we have

$$\frac{m}{n}v^m = \frac{m}{n}.(1+x)^{\frac{m}{n}} = a+2bx+3cx^2+4dx^3+5ex^4, \text{ \&c.}$$

$$ax+2bx^2+3cx^3+4dx^4, \text{ \&c.}$$

But (equation *A*,)

$$\frac{m}{n}.(1+x)^{\frac{m}{n}} = \frac{m}{n}+\frac{m}{n}ax+\frac{m}{n}bx^2+\frac{m}{n}cx^3+\frac{m}{n}dx^4, \text{ \&c.}$$

Hence by equating the homologous co-efficients,

$$a=\frac{m}{n},\ b=a.\frac{\frac{m}{n}-1}{2}=\frac{m}{n}.\frac{m-n}{2n};\ c=b.\frac{\frac{m}{n}-2}{3}=$$

$$\frac{m}{n}.\frac{m-n}{2n}.\frac{m-2n}{3n};\ d=c.\frac{\frac{m}{n}-3}{4}=\frac{m}{n}.\frac{m-n}{2n}.\frac{m-2n}{3n}.\frac{m-3n}{4n}$$

Hence, $(a+b)^{\frac{m}{n}} = a^{\frac{m}{n}}.(1+\frac{b}{a})^{\frac{m}{n}} =$

$$a^{\frac{m}{n}}.\{1+\frac{m}{n}.\frac{b}{a}+\frac{m}{n}.\frac{m-n}{2n}.\frac{b^2}{a^2}+\frac{m}{n}.\frac{m-n}{2n}.\frac{m-2n}{3n}.\frac{b^3}{a^3}, \text{ \&c.}\}$$

93. In the preceding investigation m has been supposed to be affirmative: if we now take m negative, and make the same assumptions as before, we have

$$v^{-m}-w^{-m} = \frac{1}{v^m}-\frac{1}{w^m} = \frac{w^m-v^m}{v^m.w^m} = \frac{1}{v^m.w^m}.\overline{v^m-w^m} =$$

$$-\frac{1}{v^m.w^m}.(v-w).\{v^{m-1}+v^{m-2}w+v^{m-3}w^2. \quad w^{m-1}\} =$$

$$(x-y).\{a+b(x+y)+c.(x^2+xy+y^2)+$$

$$d(x^3+x^2y+xy^2+y^3)+, \text{ \&c.}\} =$$

$$(v-w).\{v^{n-1}+v^{n-2}w. \quad w^{n-1}\}$$

$$\{a+b(x+y)+c(x^2+xy+y^2)+, \text{ \&c.}\}$$

Whence, dividing by $n.\overline{v-w}$, and assuming $x=y$,

$$-\frac{1}{v^{2n}}\times\frac{mv^{m-1}}{n}=v^{n-1}\{a+2bx+3cx^2+4dx^3+, \&c.\}$$

$\therefore$ Multiplying by v, and putting $1+x$ for v^n,

$$-\frac{m}{n}.v^{-n}=\overline{1+x}\{a+2bx+3cx^2+4dx^3+, \&c.\}=$$

$$a+2bx+3cx^2+4dx^3, \&c.$$

$$ax+2bx^2+3cx^3, \&c.$$

Now multiplying equation A by $\frac{-m}{n}$, and equating the homologous co-efficients,

$$a=\frac{-m}{n},\ b=\frac{-m}{n}\times\frac{-m-n}{2n},$$

$$c=\frac{-m}{n}\times\frac{-m-n}{2n}\times\frac{-m-2n}{3n}, \&c.$$

Whence it appears that the formula is correct whether the index be positive or negative. This is Newton's celebrated binomial theorem.

94. By the aid of this theorem, a binomial may be raised to any power, or evolved to any root by simple substitution.

Several examples of the use of this theorem, when the index is a whole positive number, are given in section second.

1. Required to develope $\sqrt[3]{\frac{a^5}{(a+x)^2}}$ in a series.

$$\sqrt[3]{\frac{a^5}{(a+x)^2}}=\sqrt[3]{\frac{a^3}{(1+\frac{x}{a})^2}}=a(1+\frac{x}{a})^{-\frac{2}{3}}$$

Here $m=-2$, $n=3$.

$$\therefore a(1+\frac{x}{a})^{-\frac{2}{3}} a.\{1-\frac{2}{3}.\frac{x}{a}+\frac{2}{3}.\frac{-2-3}{6}.\frac{x^2}{a^2}+\frac{2}{3}.\frac{-2-3}{6}$$

$$\frac{-2-6}{9}.\frac{x^3}{a^3}+\frac{2}{3}.\frac{-2-3}{6}.\frac{-2-6}{9}.\frac{-2-9}{12}.\frac{x^4}{a^4}, \text{ \&c.}\} =$$

$$a-\frac{2x}{3}+\frac{5x^2}{9a}-\frac{40x^3}{81a^2}+\frac{110x^4}{243a^3}-, \text{ \&c.}$$

2. What is the value of $\sqrt[3]{a^3+b}$ in a series?

Ans. $a+\frac{b}{3a^2}-\frac{b^2}{9a^5}+\frac{5b^3}{81a^8}-\frac{10b^4}{243a^{11}}$, &c.

3. What is the value of $\sqrt{\frac{1}{a^2-x^2}}$ in a series?

Ans. $\frac{1}{a}+\frac{x^2}{2a^3}+\frac{3x^4}{8a^5}+\frac{5x^6}{16a^7}+\frac{35x^8}{128a^9}$, &c.

4. Required the value of $(a-b)^{\frac{2}{5}}$ in a series.

Result, $a^{\frac{2}{5}}.\{1-\frac{2b}{5a}-\frac{3b^2}{25a^2}-\frac{8b^3}{125a^3}$

$$-\frac{26b^4}{625a^4}-\frac{468b^5}{15625a^5}, \text{ \&c.}\}$$

5. Required $\sqrt{\frac{a+x}{a-x}}$ in a series.*

Result, $1+\frac{x}{a}+\frac{x^2}{2a^2}+\frac{x^3}{2a^3}+\frac{3x^4}{8a^4}+\frac{3x^5}{8a^5}$, &c.

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* This quantity is reducible to

$$\sqrt{\frac{(a+x)^2}{a^2-x^2}}=\sqrt{\frac{(1+\frac{x}{a})^2}{1-\frac{x^2}{a^2}}}=(1+\frac{x}{a}).(1-\frac{x^2}{a^2})^{-\frac{1}{2}}$$

6. What is $(\frac{a}{b-x})^{\frac{3}{5}}$ in a series?

Ans. $(\frac{a}{b})^{\frac{3}{5}}.\{1+\frac{3x}{5b}+\frac{12x^2}{25b^2}+\frac{52x^3}{125b^3}+\frac{234x^4}{625b^4}$, &c.$\}$

REVERSION OF SERIES.

95. Given $z=x+\frac{x^2}{2}+\frac{x^3}{2.3}+\frac{x^4}{2.3.4}+\frac{x^5}{2.3.4.5}$ &c. to infinity, to find x in terms of z.

Assume $x=az+bz^2+cz^3+dz^4+$, &c.

Then $\frac{x^2}{2}= \frac{a^2z^2}{2}+abz^3+acz^4$, &c.
$\quad\quad\quad

Let $z=ax+bx^3+cx^5+dx^7$

Then
$$\left.\begin{aligned} -\frac{z^3}{2.3} &= -\frac{a^3}{2.3}x^3 - \frac{a^2b}{2}x^5 - \frac{ab^2}{2}x^7. \\ &\quad -\frac{a^2c}{2}x^7. \\ \frac{z^5}{2.3.4.5} &= \frac{a^5}{2.3.4.5}x^5 + \frac{a^4b}{2.3.4}x^7. \\ -\frac{z^7}{2.3.4.5.6.7} &= -\frac{a^7}{2.3.4.5.6.7}x^7. \\ -x. & \end{aligned}\right\} = 0.$$

$$\therefore a=1,\ b=\frac{1}{2.3},\ c=\frac{1}{2.3.2}-\frac{1}{2.3.4.5}=\frac{1.3}{2.4.5}$$

$$d=\frac{1}{2.4.9}+\frac{1.3}{2.8.5}-\frac{1}{4.9.4}+\frac{1}{2.3.4.5.6.7}=\frac{1.3.5}{2.4.6.7}$$

Hence, $z=x+\frac{x^3}{2.3}+\frac{1.3x^5}{2.4.5}+\frac{1.3.5x^7}{2.4.6.7}+\frac{1.3.5.7x^9}{2.4.6.8.9}$, &c.

3. Required to revert the general series $x=ay+by^2+cy^3+dy^4$, &c.

Assume $y=Ax+Bx^2+Cx^3+Dx^4$, &c. Then

$$\left.\begin{aligned} ay &= Aax+Bax^2+Cax^3+Dax^4, \text{ &c.} \\ by^2 &= bA^2x^2+2bABx^3+(B^2b+2ACb)x^4, \text{&c.} \\ cy^3 &= cA^3x^3+3A^2Bcx^4, \text{ &c.} \\ dy^4 &= dA^4x^4, \text{ &c.} \\ -x. & \end{aligned}\right\} = 0.$$

$\therefore$ (art. 89,) $A=\frac{1}{a}$, $B=-\frac{b}{a^3}$, $C=\frac{2b^2-ac}{a^5}$

$$D=\frac{5b^3-5abc+a^2d}{a^7}$$

SUMMATION OF SERIES.

96. Infinite series are sometimes of such nature, that a quantity can be found, to which the series continually approximates, and to which, without attaining perfect equality, it arrives more nearly than by any assignable difference. The quantity to which the series, by continued extension, thus approximates, is called the sum of the infinite series. Thus we say, .3333, &c. to infinity $=\frac{1}{3}$; for no number less than $\frac{1}{3}$ can be assigned, which the series may not, by extension, be made to exceed.

Required the sum of $1+x+x^2+x^3$ to infinity, x being supposed $\angle 1$.

Put $y=1+x+x^2+x^3$, to infinity.

$$\therefore xy=x+x^2+x^3, \text{ \&c. to infinity.}$$

$$\text{And } y-xy=1. \quad \therefore y=\frac{1}{1-x}$$

2. Required the sum of $1+2x+3x^2+4x^3+5x^4+$, &c. to infinity, supposing $x\angle 1$.*

Put $y=1+2x+3x^2+4x^3+5x^4$, &c.

$$\therefore -2xy=-2x-4x^2-6x^3-8x^4, \text{ \&c.}$$

$$\text{And } x^2y=x^2+2x^3+3x^4, \text{ \&c.}$$

Adding these three equations together

$$y-2xy+x^2y=1.$$

$$\therefore y=\frac{1}{1-2x+x^2}=\frac{1}{(1-x)^2}$$

* The character $>$ is used to express inequality, the opening being presented to the greater quantity.

3. Required the sum of $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}$, &c. to infinity.

Assume $x=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+$, &c. to infinity.

Then $x-1=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}$, &c.

Subtract the latter from the former, and

$1=\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}$, &c. the sum required.

4. Required the sum of $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}$, to infinity.

By last example, $1=\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}$, &c.

Subtract $\frac{1}{1.2}$. $\therefore \frac{1}{2}=\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\frac{1}{5.6}$, &c.

By subtraction $\frac{1}{2}=\frac{2}{1.2.3}+\frac{2}{2.3.4}+\frac{2}{3.4.5}+\frac{2}{4.5.6}$, &c.

Whence, $\frac{1}{4}=\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\frac{1}{4.5.6}$, &c. the sum required.

The student who desires to pursue this subject, may consult Wood's Algebra, Article 411, &c.

DIFFERENTIAL METHOD.

97. In any series of quantities, a, b, c, d, e, &c., if each term be subtracted from the next following one, and each term of the series of differences be taken from the next, and so on, the following series will be obtained.

1st differences, $b-a$, $c-b$, $d-c$, $e-d$, &c.

2d diff. $c-2b+a$, $d-2c+b$, $e-2d+c$, &c.

3d diff. $d-3c+3b-a$, $e-3d+3c-b$, &c.

4th diff. $e-4d+6c-4b+a$, &c.

Or these, 1st diff. $-a+b$, $-b+c$, $-c+d$, &c.

2d diff. $a-2b+c$, $b-2c+d$, &c.

3d diff. $-a+3b-3c+d$, $-b+3c-3d+e$, &c.

4th diff. $a-4b+6c-4d+e$,

5th diff. $-a+5b-10c+10d-5e+f$, &c.

By a little attention to these expressions, and a comparison of their formation with the powers of a binomial, we perceive that if A be taken to denote the first term of any (the nth) order of differences,

$$\pm A = a - nb + n.\frac{n-1}{2}c - n.\frac{n-1}{2}.\frac{n-2}{3}d, \text{ &c.}$$

the sign + being used when n is an even number, and — when n is odd.

If the differences of any order vanish, any one of the terms may be found by means of the others.

Suppose the 4th difference $a-4b+6c-4d+e=0$, and e was not known, we should find,

$$c = \frac{4.\overline{b+4d} - \overline{a+e}}{6}.*$$

98. Let D_1, D_2, D_3, D_4, &c. denote the first terms of the

* By this method the computers of the Nautical Almanac verify their calculations of the moon's longitude, latitude, &c.

1st, 2d, 3d, 4th, &c. orders of differences; viz. $D_1 = -a+b$, $D_2 = a-2b+c$, $D_3 = -a+3b-3c+d$, $D_4 = a-4b+6c-4d+e$, &c.; then we find $b = a+D_1$, $c = a+2D_1+D_2$, $d = a+3D_1+3D_2+D_3$, $e = a+4D_1+6D_2+4D_3+D_4$.

Hence, the $n+1$th term of the series

$$= a+nD_1+n.\frac{n-1}{2}D_2+n.\frac{n-1}{2}.\frac{n-2}{3}D_3, \text{ &c.}$$

Consequently the nth term

$$= a+\overline{n-1}.D_1+\overline{n-1}.\frac{n-2}{2}D_2+\overline{n-1}.\frac{n-2}{2}.\frac{n-3}{3}D_3, \text{ &c.}$$

1. Required the nth term of the series of odd numbers, 1, 3, 5, 7, &c.

Here the 1st diff. are 2, 2, &c.; 2d diff. 0.

$\therefore$ the nth term $= 1+2.\overline{n-1} = 2n-1$.

99. Required the sum of n terms of the series a, b, c, d, &c.

This is manifestly the same as the $\overline{n+1}$th term of the following:

$$0,\ 0+a,\ 0+a+b,\ 0+a+b+c, \text{ &c.}$$

Hence by the last article, the required sum

$$= na+n.\frac{n-1}{2}D_1+n.\frac{n-1}{2}.\frac{n-2}{3}D_2+n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}D_3, \text{ &c.}$$

1. Required the sum of n terms of the square numbers, 1, 4, 9, 16, &c.

Here $a=1$, $D_1=3$, $D_2=2$, $D_3=0$.

$\therefore$ the sum

$$=n+n.\frac{n-1}{2}.3+n.\frac{n-1}{2}.\frac{n-2}{3}.2=\frac{n.\overline{n+1}.\overline{2n+1}}{1.2.3}$$

2. Required the sum of the cube numbers, 1, 8, 27, 64, &c. continued to n terms.

Ans. $(n.\frac{n+1}{2})^2$

3. Required the sum of 25 terms of the series 1, 3, 5, 7, &c. Ans. 625.

4. Required the sum of 15 terms of the series 1, 16, 81, 256, 625, 1296, &c. Ans. 178312.

When the differences at length vanish, any term of the series, or the sum of any number, may be *accurately determined* by the methods used in this and the preceding articles: when the differences become small, but do not vanish, a near approximation can be made.

100. In article 98, the number n is supposed to be an integer, in which case, if the differences vanish, the rule is demonstrably correct; the same formula is, however, applied to the case where n is fractional.

Suppose a series p, q, r, s, &c. of equidifferent quantities and another series, a, b, c, d, &c., such that the terms of the latter shall be similar functions of the correlative terms of the former. It is proposed to find a term y, in the latter series, corresponding to v in the former; $v-p$ being given.

Let $q-p : v-p :: 1 : x$, and take as before, D_1, D_2, D_3, &c. the first of the 1st, 2d, 3d, &c. differences.

Then assuming the series, (art. 98,)

$$y=a+xD_1+x.\frac{x-1}{2}D_2+x.\frac{x-1}{2}.\frac{x-2}{3}D_3, \text{ &c.}$$

This gives, when $x=0$, $y=a$, $x=1$, $y=a+D_1$, $x=2$,

$y=a+\underset{1}{2D}+\underset{2}{D}$, &c.; but these expressions equal b, c, &c. as they ought to do. This formula is, therefore, presumed to be universally true, whether x is integral or fractional.

Required to find the $6\frac{1}{2}$th term in the series 1, 4, 9, 16.

Here $x=\frac{11}{2}$, $a=1$, $\underset{1}{D}=3$, $\underset{2}{D}=2$, $\underset{3}{D}=0$.

$$\therefore y=1+\frac{11}{2}\times\frac{3}{1}+\frac{11}{2}\times\frac{9}{4}\times\frac{2}{1}=\frac{169}{4}$$

2. Let a, b, c, d, be arcs of a great circle intercepted between a fixed star and the moon's centre at noon and midnight of two successive days, it is required to find the distance at 15 hours from the first noon.

Here $12:15::1:x=\frac{5}{4}$

$$\therefore \text{dist. } (y)=a+\frac{5}{4}\underset{1}{D}+\frac{5}{4}\times\frac{1}{8}\underset{2}{D}+\frac{5}{4}\times\frac{1}{8}\times\frac{-3}{12}\underset{3}{D}=$$

$$a+\frac{5}{4}\underset{1}{D}+\frac{5}{32}\underset{2}{D}-\frac{5}{128}\underset{3}{D}$$

CONSTRUCTION OF LOGARITHMS.

101. From article 85, it is obvious that, a being any number, if $a^n=N$, and $a^m=M$, $a^{n+m}=NM$, and $a^{n-m}=\frac{N}{M}$. If, therefore, all the numbers used in calculation were expressed in powers of a, multiplication and division might be performed by the addition and subtraction of the exponents of a. These exponents, thus employed,

are termed *logarithms*. When $a^n=N$, n is the logarithm of N.

Let x be such number, that a^x may denote any proposed number.

Put $a=1+b$, then (art. 91, 92, 93,)

$$(1+b)^x=1+xb+x.\frac{x-1}{2}b^2+x.\frac{x-1}{2}.\frac{x-2}{3}b^3+x.\frac{x-1}{2}.$$

$$\frac{x-2}{3}.\frac{x-3}{4}b^4, \text{ \&c.}=1+bx+b^2\left\{\frac{x^2}{2}-\frac{x}{2}\right\}+$$

$$b^3\left\{\frac{x^3}{6}-\frac{x^2}{2}+\frac{x}{3}\right\}+b^4\left\{\frac{x^4}{24}-\frac{x^3}{4}+\frac{11x^2}{24}-\frac{x}{4}\right\} \text{ \&c.}$$

If, therefore, we collect the terms containing the simple power of x, those containing x^2, and denote the above equation thus,

$$(1+b)^x=1+Ax+Bx^2+Cx^3+Dx^4, \text{ \&c.}$$

We readily find that

$$A=b-\frac{b^2}{2}+\frac{b^3}{3}-\frac{b^4}{4}+\frac{b^5}{5}, \text{ \&c. } (Q.)$$

If for x in the above equation we substitute $2x$, we shall have

$$(1+b)^{2x}=1+2Ax+4Bx^2+8Cx^3+16Dx^4, \text{ \&c.}$$

Squaring the same equation,

$$(1+b)^{2x}=1+2Ax+2Bx^2+2Cx^3+2Dx^4. \text{ \&c.}$$
$$A^2x^2+2ABx^3+2ACx^4, \text{ \&c.}$$
$$B^2x^4, \text{ \&c.}$$

Comparing the co-efficients of like powers of x.

$$B=\frac{A^2}{2}, C=\frac{A^3}{2.3}. D=\frac{A^4}{2.3.4}, \text{ \&c.}$$

$$\therefore a^x=(1+b)^x=1+Ax+\frac{A^2x^2}{2}+\frac{A^3x^3}{2.3}+\frac{A^4x^4}{2.3.4}, \text{ \&c. } (R.)$$

In like manner,

$$\frac{1}{(1-b)^x}=(1-b)^{-x}=1+xb+x.\frac{x+1}{2}b^2+x.\frac{x+1}{2}.\frac{x+2}{3}b^3, \text{ \&c.}$$

$$=1+Ax+\frac{A^2x^2}{2}+\frac{A^3x^3}{2.3}+\frac{A^4x^4}{2.3.4}, \text{ \&c. } (S.) \text{ in which}$$

$$A=b+\frac{b^2}{2}+\frac{b^3}{3}+\frac{b^4}{4}, \text{ \&c. } (I.)$$

102. Let $1+y$ denote a number whose logarithm is required; then $1+y=(1+b)^x$.

$\therefore (1+y)^z=(1+b)^{xz}$, z being any number whatever.

Here x is the logarithm of $(1+b)^x$, or of a^x, or $1+y$.

But $(1+y)^z=1+Vz+\frac{V^2z^2}{2}+\frac{V^3z^3}{2.3}$, &c. (*R.* 101.)

V being $=y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{4}+\frac{y^5}{5}$, &c.(*Q.*101.)

Also $(1+b)^{xz}=1+Axz+\frac{A^2x^2z^2}{2}+\frac{A^3x^3z^3}{2.3}$,&c.(*R.*101.)

Whence by comparing the co-efficients of z,

$$Ax=V, \text{ and } x=\frac{V}{A}=\frac{y-\frac{1}{2}y^2+\frac{1}{3}y^3-\frac{1}{4}y^4+\frac{1}{5}y^5}{A}, \text{ \&c.} =$$

$$M.\{y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{4}+\frac{y^5}{5}, \text{ \&c.}\} \text{ putting } M=\frac{1}{A}$$

Again, putting $\frac{1}{1-y}=$ a number whose logarithm is required, and making $\frac{1}{1-y}=(1+b)^v$, we find

$$v \text{ (the log. of } (1+b)^v \text{ or } \frac{1}{1-y}) =$$

$M \times \{y + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4} + \frac{y^5}{5}$, &c.$\}$ by a process exactly analagous to the former.

Now the logarithm of $\frac{1+y}{1-y}$ (or of $1+y \times \frac{1}{1-y}$ = logarithm of $(1+y)$ + logarithm of $\frac{1}{1-y}$) =

$$x+v=2M\{y+\frac{y^3}{3}+\frac{y^5}{5}+\frac{y^7}{7}, \text{ \&c.}\} \ (U.)$$

As is manifest by adding together the values of x and v above obtained.

103. As A, and consequently $\frac{1}{A}$ or M, is given in terms of b, $(a-1)$ and a may be assumed of any value whatever, (art. 101,) it follows that M is not limited to one particular value.

If the value of a be assumed, M may be thence determined, or if we take any number at pleasure as the value of M, a will be limited; but in this case the formulæ U may be applied to the construction of logarithms, without computing the value of a. M is termed the modulus, and a the radix of the system. Instead of fixing the value of M, or a, by arbitrary assumption, we may take 1 as the logarithm of any proposed number, and thence deduce the value of M.

Since $a^0=1$, the logarithm of 1 must, in every system, $=0$. If we make $1=$ logarithm of 10, the radix will be 10, and M might have been found from equation I, (art. 102,) by calling b 9, if that series or its reciprocal had been a converging one. As that is not the case, a different expedient must be used.

Let $\frac{p+1}{p}$ denote a number whose logarithm is to be

found, and put $\frac{1+y}{1-y}=\frac{p+1}{p}$; then $y=\frac{1}{2p+1}$, this value substituted for y in equation U, (art. 102,) will make the series always converge, and more rapidly the greater the value of p.

Now since $\frac{p+1}{p}\times\frac{p}{1}=p+1$, it is evident that the log. of $\overline{p+1}$ = log. p + log. of $\frac{p+1}{p}$.

First, let $p=1$, then $\frac{p+1}{p}=2$, and $\frac{1}{2p+1}=\frac{1}{3}$, also log. $p=0$.

$$\frac{1}{3}=.3333333333$$

$$\frac{1}{3^3}=\frac{1}{9.3}=\ 370370370$$

$$(\tfrac{1}{3})^5=\tfrac{1}{9}\text{ of }(\tfrac{1}{3})^3=\ 41152263$$

$$(\tfrac{1}{3})^7=\tfrac{1}{9}\text{ of }(\tfrac{1}{3})^5=\ 4572474\ (A.)$$

$$(\tfrac{1}{3})^9=\tfrac{1}{9}\text{ of }(\tfrac{1}{3})^7=\ 508053$$

$$\tfrac{1}{3})^{11}=\tfrac{1}{9}\text{of }(\tfrac{1}{3})^9=\ 56450\ (B.)$$

$$(\tfrac{1}{3})^{13}=\ 6272\ (C.)$$

$$(\tfrac{1}{3})^{15}=\ .\ 697$$

$$(\tfrac{1}{3})^{17}=\ 77\ (D.)$$

$$\frac{1}{3} = .3333333363$$

$$\frac{1}{3.3^3} = \quad 123456790$$

$$\frac{1}{5.3^5} = \quad 8230453$$

$$\frac{1}{7.3^7} = \quad 653211$$

$$\frac{1}{9.3^9} = \quad 56450$$

$$\frac{1}{11.3^{11}} = \quad 5132$$

$$\frac{1}{13.3^{13}} = \quad 483$$

$$\frac{1}{15.3^{15}} = \quad 46$$

$$\frac{1}{17.3^{17}} = \quad 4$$

$$.3465735902$$

Hence, log. of $2 = .6931471804M$, and log. of 8 (or 2^3) $= 2.0794415412M$.

Second, let $p = 4$, then $\frac{1}{2p+1} = \frac{1}{9}$

Hence, to find the log. of $\frac{5}{4}$

$$\frac{1}{9} \quad =.1111111111$$

$$\frac{1}{3}\text{ of }(\frac{1}{9})=A= \quad 4572474$$

$$\frac{1}{5}\text{ of }(\frac{1}{9})^2=.6B= \quad 33870$$

$$\frac{1}{7}\text{ of }(\frac{1}{9})^7=\frac{C}{21}= \quad 294$$

$$\frac{1}{9}\text{ of }(\frac{1}{9})^9=\frac{D}{27} \quad 3$$

$$.1115717752$$

Whence, log. of $\frac{5}{4}=.2231435504M.$

Consequently the log. of $(\frac{8}{1}\times\frac{5}{4}=)10$

$$=2.3025850914M.$$

But log. of 10=1.

Hence, $M=\frac{1}{2.3025850914}=.4342944822.$

Substituting for M the number last found, we obtain the log. of 2=.30102999599, and thence the log. of 4, 8, 16, 32, &c. may be had by simple multiplication. Also the log. of 5= log. of 10— log. of 2=.69897000401; whence may be found the logarithms of all the powers of 5.

The numerical value of M being thus ascertained, if we make p denote a number whose log. is known, we have $\log.\overline{p+1}=\log.p+\log.\frac{p+1}{p}=\log.p+$

$$\frac{2M}{2p+1}\times\frac{A}{3.(2p+1)^2}+\frac{B}{5.(2p+1)^4}+\frac{C}{7.(2p+1)^6},\text{ &c.}$$

Where A, B, C, &c. denote the preceding term exclusive of the divisor, 3, 5, &c., and thus the logarithms of all the prime numbers may be computed. But as the above series, when the number p is small, converges but slowly, and therefore, requires a considerable number of terms to be used, the labour may be abridged by proper expedients, a few of which are subjoined.

Let the log. of 3 be required; put $p=80$, then

$$\frac{1}{2p+1}=\frac{1}{161}$$

And log. of $\frac{p+1}{p}=\frac{2M}{161}+\frac{A}{3\times 25921}=$ .0053950319

To which add log. of $80=*1+3$ log. of $2=$ 1.9030899880

Hence, log. of 81, or 4 log. of 3= 1.9084850199

And log. 3=.4771212549.

Again, let the log. of 7 be required; put $p=27$, then

$$\frac{1}{2p+1}=\frac{1}{55}$$

And log. $\frac{28}{27}=\frac{2M}{55}+\frac{A}{9075}$ = .0157942671

Adding log. of 3^3, or of 27=1.4313637647

Log. of 28=1.4471580318

Subtracting log. of 4, .6020599920

Log. of 7= .8450980398

* As the logarithms of 10 and its powers are wholly integral, it is manifest the logarithm of a number is changed only in the integral part, by varying the position of the decimal point in the number itself.

In general, suppose $x+2$ to denote a prime number whose logarithm is required, those of the inferior even numbers being known.

Since $\frac{(x-1)^2\times x+2}{(x+1)^2\times(x-2)}=\frac{x^3-3x+2}{x^3-3x-2}$, where both terms are divisible by 4, because $x-1$, and $x+1$ are even numbers, it follows that this fraction in its lowest terms, may be expressed by $\frac{p+1}{p}$, and therefore the log. of

$$\frac{(x-1)^2.(x+2)}{(x+1)^2.(x-2)}$$

be obtained from the formulæ above given; to which adding 2 log. $(x+1)$+ log. $(x-2)$—2 log. $(x-1)$ the result will be the log. of $x+2$.

Let the log of 13 be required. Here $x=11$, and our fraction becomes

$$\frac{10^2.13}{12^2.9}=\frac{1300}{1296}=\frac{325}{324}$$

Hence, $p=324$, and $\frac{1}{2p+1}=\frac{1}{649}$

Whence, log. $\frac{325}{324}$= .0013383507. To which add 4 log. 3 + log. of 4—2, log. of 5= .1126050039

Log. of 13=1.1139433546

In this manner the logarithms may be derived from those already obtained, to any proposed extent. The differential method may be advantageously applied, to the completion of a logarithmic table. But the labour of those computations, being already finished, and not likely to be renewed, a further elucidation of the subject is deemed unnecessary.

Section X.

SURDS.

104. It has been remarked, (note, art. 36.) that the even roots of negative quantities are impossible; hence whenever, in the solution of a problem, the square root of a negative quantity appears in the result, such result is impossible, or imaginary. But instead of abandoning, as hopeless, every example in which such expressions appear, they are found conformable to the general principles of the science, and sometimes connected with the most refined analytical processes.

Impossible roots may be introduced into a problem in two ways, quite distinct from each other. First, by admitting incompatible assumptions into the data of the problem; in which case the impossible root serves to detect that incompatibility; and its appearance or disappearance marks the limits of the problem.

Thus in the equation $x^2-2ax=-b$, we have (by art. 62,) $x=a\pm\sqrt{a^2-b}$ as the general expression of the value of x. If now $a^2>b$,* a^2-b is positive, and therefore, $\sqrt{a^2-b}$, and consequently x, a possible quantity. But if $a^2<b$, a^2-b is negative; and therefore, $\sqrt{a^2-b}$ impossible; whence x is, in this case, an imaginary quantity. Here the terms of the original equation are compatible, whenever a^2 is equal to, or greater than b; but incompatible when $a^2<b$; for $x^2-2ax=-b$ is equivalent to $\overline{2a-x}.x=b$, but $\overline{2a-x}.x$ cannot be greater than a^2.

Put $x=a\pm c$, then $2a-x=a\mp c$, and $\overline{2a-x}.x=a^2-c^2$.

When $c=0$, $x=a$, or $\overline{2a-x}.x=(b=)a^2$; this is, therefore, the limit of b.

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* $a>b$ is read a is greater than b, and $a<b$, a is less than b.

Suppose $x+\frac{1}{x}=2a$, to find x.

Multiplying by x, $x^2+1=2ax$, or $x^2-2ax=-1$.

Whence, $x=a\pm\sqrt{a^2-1}$, which value of x is imaginary when $a<1$. Hence the least possible sum of a number and its reciprocal is 2.*

105. Impossible roots may be sometimes obtained, when the data are compatible, by the admission of inconsistent suppositions into the solution.

Suppose we have $x^2+ny^2=a$, and $xy=b$, to find x and y; a, b, and n being any given numbers.

To and from the first equation add and subtract $2\sqrt{n}$ times the second, whence, $x^2+2xy\sqrt{n}+ny^2=a+2b\sqrt{n}$, and $x^2-2xy\sqrt{n}+ny^2=a-2b\sqrt{n}$; and by evolution,

$$x+y\sqrt{n}=\sqrt{a+2b\sqrt{n}}, \quad (M.)$$

$$x-y\sqrt{n}=\sqrt{a-2b\sqrt{n}}, \quad (N.)$$

$$\therefore x=\frac{\sqrt{a+2b\sqrt{n}}+\sqrt{a-2b\sqrt{n}}}{2}$$

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* These principles may be applied to determine the maxima and minima of geometrical quantities. Let $b=$ the base, $a=$ the altitude of a plane triangle; $x=$ altitude of its inscribed rectangle, then $bx-\frac{bx^2}{a}=$ the area of the rectangle, which put $=c$; whence we find $x=\frac{a}{2}\pm\sqrt{(\frac{a}{4}-\frac{ac}{b})}$ which is impossible when $c>\frac{ab}{4}$

If $c=\frac{ab}{4}$, $x=\frac{a}{2}$, this is, therefore, the greatest possible value of x; and the greatest rectangle is $\frac{ab}{4}=\frac{1}{2}$ the triangle.

And $y=\frac{\sqrt{a+2b\sqrt{n}}-\sqrt{a-2b\sqrt{n}}}{2\sqrt{n}}$

Which are general expressions for the values of x and y.

If now, we take $n=-1$, or $x^2-y^2=a$, the above expressions become,

$$x=\frac{\sqrt{a+2b\sqrt{-1}}+\sqrt{a-2b\sqrt{-1}}}{2}$$

$$y=\frac{\sqrt{a+2b\sqrt{-1}}-\sqrt{a-2b\sqrt{-1}}}{2\sqrt{-1}}$$

These values, though expressed by imaginary quantities, are real ones. For putting -1 instead of n, in equations M, N, and multiplying, we have

$$x^2+y^2=\sqrt{a^2+4b^2}, \ (P.)$$

But from the first equation, $x^2-y^2=a$.

Consequently, by addition and subtraction,

$$2x^2=\sqrt{a^2+4b^2}+a.$$

And $2y^2=\sqrt{a^2+4b^2}-a$,

$$\text{or } x=\left(\frac{\sqrt{a^2+4b^2}+a}{2}\right)^{\frac{1}{2}}; \ y=\left(\frac{\sqrt{a^2+4b^2}-a}{2}\right)^{\frac{1}{2}}$$

The same conclusions may be obtained by squaring the values of x and y first found, and extracting the square roots of the results.

The impossible surd $\sqrt{-1}$, was evidently introduced into the result, by adopting a process in the general solution, which was not applicable to the particular equation $x^2-y^2=a$; yet the values of x and y, when cleared of imaginary surds, are the true ones; as may be shown by a different solution of the problem.

To the square of the first equation adding 4 times the square of the second, $x^4+2x\ y^2+y^2=a^2+4b^2$.

Whence by evolution, $x^2+y^2=\sqrt{a^2+4b^2}$, the same as equation P, deduced from imaginary surds.

106. From what is shown in the foregoing article, we readily infer, that when, in the solution of a problem, the value of the quantity sought appears in terms of imaginary surds, we are not thence immediately to conclude, that the data are inconsistent; as the adoption of an inapplicable process may produce such a result, yet in this case the imaginary surd may be eliminated by the use of proper expedients. When, however, the data are inconsistent, no analytical address can clear the final equation of its impossible quantities.

Imaginary surds differ from real ones in this important particular. Real surds, however complex, admit of an approximation to their value; but imaginary surds admit of no approximation, and must either be eliminated, when practicable, or remain the intractable indications of incongruous assumptions.

The following cases exhibit the most useful applications of algebra to surd quantities.

Case 1.

107. *To reduce a rational quantity to the form of a surd.*

Raise the given quantity to the power denoted by the index of the surd, and to this power apply the radical sign or index proposed.

EXAMPLES.

1. Reduce 5 to the form of a square root, and a to that of a 4th root.

$$5=\sqrt{5^2}=\sqrt{25}, \text{ and } a=(a^4)^{\frac{1}{4}}=\sqrt[4]{a^4}.$$

2. Reduce 3 to the form of a cube root.

Result, $\sqrt[3]{27}$.

3. Express $-\frac{1}{3}a$ in the form of a cube root.

Result, $(-\frac{1}{27}a^3)^{\frac{1}{3}}$.

4. Reduce $2\sqrt{5}$ to the form of a square root.

Result, $\sqrt{20}$.

5. Reduce $3\sqrt{2}$ to the form of a 4th root.

Result, $(324)^{\frac{1}{4}}$.

6. Express $a+b$ in the form of a square root.

Result, $\sqrt{a^2+2ab+b^2}$.

7. Express $\frac{1+\sqrt{5}}{2}$ in the form of a quadratic surd.

Result, $\sqrt{\frac{6+2\sqrt{5}}{4}}$

Case 2.

108. *To reduce radical quantities, having different indices, to other equivalent quantities with a common radical sign.*

Reduce the fractional exponents to a common denominator, involve the given quantities to the powers denoted by their respective numerators; and to the results apply the reciprocal of the common denominator as the common exponent.

EXAMPLES.

1. Reduce $a^{\frac{2}{3}}$, $b^{\frac{3}{4}}$, and $c^{\frac{1}{5}}$ to equivalent quantities, having a common exponent.

$\frac{2}{3}, \frac{3}{4}, \frac{1}{5}$ reduced are equivalent to $\frac{40}{60}, \frac{45}{60}, \frac{12}{60}$

Hence, $a^{\frac{2}{3}}=(a^{40})^{\frac{1}{60}}=a^{\frac{40}{60}}$;

$b^{\frac{3}{4}}=(b^{45})^{\frac{1}{60}}=b^{\frac{45}{60}}$; $c^{\frac{1}{5}}=(c^{12})^{\frac{1}{60}}=c^{\frac{12}{60}}$.

2. Reduce $3^{\frac{1}{2}}$, and $4^{\frac{1}{3}}$ to a common index.

Result, $27^{\frac{1}{6}}$, $16^{\frac{1}{6}}$.

3. Reduce $a^{\frac{2}{3}}$, $b^{\frac{1}{5}}$, to a common radical sign.

Result, $a^{\frac{5}{15}}$, $b^{\frac{3}{15}}$.

4. Reduce $(a+x)^{\frac{1}{2}}$, $(a-x)^{\frac{1}{3}}$ to a common index.

Result, $(a^3+2ax+x^3)^{\frac{1}{6}}$, and $(a^3-3a^2x+3ax^2-x^3)^{\frac{1}{6}}$.

5. Reduce $a^{\frac{1}{3}}$, and $x^{\frac{1}{4}}$ to the common index $\frac{1}{12}$.*

Result, $(a^4)^{\frac{1}{12}}$, $(x^3)^{\frac{1}{12}}$.

6. Reduce $4^{\frac{1}{3}}$, and $5^{\frac{1}{2}}$ to a common index.

Result, $(256^4)^{\frac{1}{4}}$, and $25^{\frac{1}{4}}$.

Case 3.

109. *To reduce surds to their most simple terms.*

Resolve the surd, if possible, into two factors, one of which shall be the greatest power it contains. Extract the root, and thereto annex the other factor with its proper radical sign.

EXAMPLES.

1. Reduce $\sqrt[3]{250}$ to its simplest terms.

$250=125\times2=5^3\times2$. $\therefore \sqrt[3]{250}=5\sqrt[3]{2}$.

2. Reduce $\sqrt{32}$ to its simplest terms.

Result, $4\sqrt{2}$.

---

* When the common index, to which the fractional exponents are to be reduced is given, divide each of the given exponents by that common index, and involve the given quantities to the powers indicated by the quotients.

3. Reduce $\sqrt{243}$ to its simplest form.

Result, $3\sqrt{3}$.

4. Reduce $\sqrt[3]{5184}$ to simple terms.

Result, $12\sqrt[3]{3}$.

5. Reduce $(144)^{\frac{1}{4}}$ to its simplest terms.

Result, $2\sqrt{3}$.

6. Reduce $(a^3b-a^3x)^{\frac{1}{3}}$ to simple terms.

Result, $a\sqrt[3]{b-x}$.

7. Reduce $\sqrt{\frac{18}{125}}$ to its simplest terms.*

Result, $\frac{3}{25}\sqrt{10}$.

8. Reduce $\sqrt[3]{\frac{8}{9}}$ to simple terms.

Result, $\frac{2}{3}\sqrt[3]{3}$.

9. Reduce $\frac{\sqrt{20}+\sqrt{12}}{\sqrt{5}-\sqrt{3}}$ to its simplest form.

Result, $8+2\sqrt{15}$.

Case 4.

110. *To add or subtract surd quantities.*

Reduce the quantities, when fractional, to a common denominator; and, when the exponents are different, to a common radical sign. Also express the surds in their simplest terms. If then the reduced surds are alike, they may be added or subtracted as other algebraic quan-

* When the given surd is fractional, the denominator may generally be made rational, without changing the value of the fraction, by multiplying both terms by proper numbers or quantities. When the denominator consists of two quadratic surds connected by the sign + or —, the proper multiplier consists of the same surds connected by the opposite sign.

tities. When the surds are unlike, the operation must be merely indicated.

EXAMPLES.

1. Required the sum and difference of $\sqrt[3]{128}$ and $\sqrt[3]{16}$.

$\sqrt[3]{128}=4\sqrt[3]{2}$, $\sqrt[3]{16}=2\sqrt[3]{2}$, $4\sqrt[3]{2}+2\sqrt[3]{2}=6\sqrt[3]{2}$,

$4\sqrt[3]{2}-2\sqrt[3]{2}=2\sqrt[3]{2}$.

2. What is the sum of $\sqrt{27}$ and $\sqrt{48}$?

Ans. $7\sqrt{3}$.

3. What is the difference between $\sqrt{50}$ and $\sqrt{18}$?

Ans. $2\sqrt{2}$.

4. Required the sum of $\sqrt[3]{56}$ and $\sqrt[3]{189}$.

Ans. $5\sqrt[3]{7}$.

5. What is the sum of $3\sqrt{a^2b}$ and $5\sqrt{16a^4b}$?

Ans. $(3a+20a^2)\sqrt{b}$.

6. What difference is there between $\sqrt{24a^3b^2}$ and $\sqrt{54ab^4}$? Ans. $(2ab-3b^2)\sqrt{6a}$.

7. Required the sum of $\sqrt[3]{\frac{1}{4}}$ and $\sqrt[3]{\frac{1}{32}}$?

Ans. $\frac{3}{4}\sqrt[3]{2}$.

8. Required the difference beetween $\frac{\sqrt{14}+\sqrt{12}}{\sqrt{7}-\sqrt{6}}$ and $\frac{\sqrt{6}-\sqrt{4}}{\sqrt{3}+\sqrt{2}}$? Ans. $8\sqrt{2}+4\sqrt{21}+4\sqrt{3}$.

CASE 5.

111. *To multiply or divide surd quantities.*

Like quantities with different exponents, are multiplied or divided by adding or subtracting their indices.

When the quantities are different reduce them, if necessary, to their equivalent ones with a common radical sign. Then the product or quotient, with the common radical sign applied, will be the quantity sought.*

EXAMPLES.

1. Required the product of $5\sqrt[3]{2}$ by $7\sqrt{3}$.

$5\sqrt[3]{2}=5\sqrt[6]{4}$, $7\sqrt{3}=7\sqrt[6]{27}$, $5\sqrt[6]{4}\times 7\sqrt[6]{27}=35\sqrt[6]{108}$.

2. Divide $\sqrt{21}+\sqrt{15}$ by $\sqrt{7}-\sqrt{5}$.

$$\frac{\sqrt{21}+\sqrt{15}}{\sqrt{7}-\sqrt{5}}=\frac{\sqrt{21}+\sqrt{15}}{\sqrt{7}-\sqrt{5}}\times\frac{\sqrt{7}+\sqrt{5}}{\sqrt{7}+\sqrt{5}}=\frac{12\sqrt{3}+2\sqrt{105}}{7-5}$$

$=6\sqrt{3}+\sqrt{105}$, the quotient required.

3. Required the product of $5\sqrt{2}$ by $7\sqrt{3}$.

Result, $35\sqrt{6}$.

4. What is the product of $3\sqrt{5}$ by $4\sqrt[3]{25}$?

Ans. 60.

5. What is the product of $\frac{5}{3}\sqrt{\frac{3}{8}}$ by $\frac{9}{10}\sqrt{\frac{2}{5}}$?

Ans. $\frac{3}{20}\sqrt{15}$.

6. What is the product of $2\sqrt[3]{14}$ and $3\sqrt[3]{4}$?

Ans. $12\sqrt[3]{7}$.

7. What is the quotient of $6\sqrt[3]{972}$ by $3\sqrt[3]{2}$?

Ans. $6\sqrt[3]{18}$.

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* In the division of surds, as well as of rational quantities, it is frequently convenient to set down the terms as a vulgar fraction, and to reduce that fraction to its simplest form.

8. What quotient will $\frac{5}{6}\sqrt{\frac{1}{50}}$ divided $\frac{3}{4}\sqrt{\frac{2}{5}}$ make?

Ans. $\frac{1}{9}\sqrt{5}$.

9. What is the product of $(a+b)^{\frac{1}{3}}$ and $(a+b)^{\frac{3}{4}}$?

Ans. $(a+b)^{\frac{13}{12}}$.

10. What is the product of $(a+2\sqrt{b})^{\frac{1}{2}}$ and $(a-2\sqrt{b})^{\frac{1}{2}}$?

Ans. $(a^2-4b)^{\frac{1}{2}}$.

11. What is the product of $3+\sqrt{-2}$ by $5-\sqrt{-2}$?

Ans. $17+2\sqrt{-2}$.

12. What is the product of $(a+\sqrt{-b})^{\frac{1}{2}}$ by $(a-2\sqrt{-b})^{\frac{1}{2}}$? Ans. $(a^2+2b-a\sqrt{-b})^{\frac{1}{2}}$.

13. Divide $\frac{4}{5}\sqrt{a}$ by $\frac{2}{3}\sqrt[3]{a}$. Quotient, $\frac{6}{5}a^{\frac{1}{6}}$.

14. Divide $3a^{\frac{1}{n}}$ by $4a^{\frac{1}{m}}$. Quot. $\frac{3}{4}a^{\frac{m-n}{mn}}$

15. Divide $5+3\sqrt{-3}$ by $7-2\sqrt{-3}$.

Quot. $\frac{17}{61}+\frac{31}{61}\sqrt{-3}$.

As involution is effected by the multiplication of equal factors, this case evidently includes the involution of surd quantities.

16. What is the 3d power of $4\sqrt{2}$?

Ans. $128\sqrt{2}$.

17. Required the nth power of $a^{\frac{1}{m}}$? Ans. $a^{\frac{n}{m}}$

18. What is the 3d power of $3+2\sqrt{-1}$?

Ans. $46\sqrt{-1}-9$.

Case 6.

112. *To extract the roots of surd quantities.*

When the quantity consists of one term only, the root is obtained by dividing the index of each quantity by the index of the power.

When the quantity is a compound one, the root may sometimes be extracted, as in article 37.

When the given quantity is a binomial, one term, at least, of which is a quadratic surd, the square root may be extracted by the following formula.

Let $\sqrt{a \pm \sqrt{b}} = x$, put $\sqrt{a^2 - b} = d$, then

$$x = \sqrt{(\frac{1}{2}a + \frac{1}{2}d)} \pm \sqrt{(\frac{1}{2}a - \frac{1}{2}d)}^*$$

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* This formula is thus obtained,

Let $\sqrt{a \pm \sqrt{b}} = \sqrt{v} \pm \sqrt{w}$.

Then $a \pm \sqrt{b} = v \pm 2\sqrt{vw} + w$.

Now, as a rational quantity cannot be equal to a surd, we must take

$$a = v + w, \ (\mathcal{N}.) \text{ and } \sqrt{b} = 2\sqrt{vw}.$$

By squaring these equations,

$$a^2 = v^2 + 2vw + w^2, \text{ and } b = 4vw.$$

Whence, by subtraction and evolution,

$$\sqrt{a^2 - b} = \sqrt{v^2 - 2vw + w^2}, \text{ or } d = v - w.$$

From this and equation $\mathcal{N}$, we have,

$$v = \frac{1}{2}a + \frac{1}{2}d, \text{ and } w = \frac{1}{2}a - \frac{1}{2}d.$$

$$x = \sqrt{v} \pm \sqrt{w} = \sqrt{(\frac{1}{2}a + \frac{1}{2}d)} \pm \sqrt{(\frac{1}{2}a - \frac{1}{2}d.)}$$

Hence, it appears that this method applies only to the case where $a^2 - b$ is a complete square.

EXAMPLES.

1. Required the square root of $16\sqrt{6}$.

$\sqrt{16}=4$, and $\sqrt{6^{\frac{1}{2}}}=6^{\frac{1}{4}}$. $\therefore \sqrt{16\sqrt{6}}=4\sqrt[4]{6}$.

2. What is the square root of 6^3?

Ans. $6^{\frac{3}{2}}$ or $6\sqrt{6}$.

3. What is the cube root of $\frac{1}{27}\sqrt{3}$?

Ans. $\frac{1}{3}\sqrt[6]{3}$.

4. What is the square root of $a^2-6a\sqrt{-b}-9b$?

Ans. $a-3\sqrt{-b}$.

5. Required the square root of $6+2\sqrt{5}$.

Here $a=6$, $\sqrt{b}=2\sqrt{5}=\sqrt{20}$.

$\therefore b=20$, and $d=\sqrt{36-20}=4$.

Whence, $(6+2\sqrt{5})^{\frac{1}{2}}=\sqrt{\frac{6+4}{2}}+\sqrt{\frac{6-4}{2}}=\sqrt{5}+1$, the root required.

6. What is the square root of $3-2\sqrt{2}$?

Ans. $\sqrt{2}-1$.

7. What is the square root of $7-2\sqrt{10}$?

Ans. $\sqrt{5}-\sqrt{2}$.

PROMISCUOUS EXAMPLES.

113. Required the difference between

$$\frac{\sqrt{5+2\sqrt{5}}}{2} \text{ and } \frac{3+\sqrt{5}}{2\sqrt{(5+2\sqrt{5})}}; \frac{\sqrt{5+2\sqrt{5}}}{2}=$$

$$\frac{5+2\sqrt{5}}{2\sqrt{(5+2\sqrt{5})}};\ \frac{5+2\sqrt{5}}{2\sqrt{(5+2\sqrt{5})}}-\frac{3+\sqrt{5}}{2\sqrt{(5+2\sqrt{5})}}=$$

$$\frac{2+\sqrt{5}}{2\sqrt{(5+2\sqrt{5})}}=\frac{\sqrt{9+4\sqrt{5}}}{2\sqrt{(5+2\sqrt{5})}}=$$ (multiplying num. and denom. by $\sqrt{5-2\sqrt{5}}$) $\frac{\sqrt{5+2\sqrt{5}}}{2\sqrt{5}}=\frac{1}{2}\sqrt{(1+\frac{2}{5}\sqrt{5})}$ the difference sought.

2. Required the sum of $5\sqrt{2}-1$, and $\frac{3\sqrt{2}+2}{5-3\sqrt{2}}$?

Ans. $8\sqrt{2}+3$.

3. What is the sum of $12\sqrt[3]{\frac{1}{4}}$ and $3\sqrt[3]{\frac{1}{32}}$?

Ans. $\frac{27}{4}\sqrt[3]{2}$.

4. Required the difference of $3\sqrt[3]{\frac{1}{3}}$ and $\sqrt[3]{72}$?

Ans. $\sqrt[3]{9}$.

5. What difference is there between $2\sqrt[3]{a^6b^4}$ and $\sqrt{8a^3b^7}$? Ans. $2ab(a-b)\sqrt[3]{b}$.

6. Required the product of $4+2\sqrt{2}$ by $2-\sqrt{2}$?

Ans. 4.

7. Multiply $a+b\sqrt{-1}$ by $a-b\sqrt{-1}$.

Product, a^2+b^2.

8. Divide $\frac{4+\sqrt{-3}}{2}$ by $\frac{1-\sqrt{-3}}{2}$.

Quotient, $\frac{1-3\sqrt{-3}}{4}$

9. Required the square root of $51-10\sqrt{2}$?

Ans. $5\sqrt{2}-1$.

10. Required the 3d power of $a-b\sqrt{-1}$.

Ans, $a^3-3a^2b\sqrt{-1}+3ab^2+b^3\sqrt{-1}$.

11. Required the square root of $7-24\sqrt{-1}$?

Ans. $4-3\sqrt{-1}$.

12. Divide a^3+b^3 by $a-b\sqrt{-1}$.

Quotient, $a+b\sqrt{-1}$.

13. Add $\frac{a-b\sqrt{-1}}{a+b\sqrt{-1}}$ to $\frac{a+b\sqrt{-1}}{a-b\sqrt{-1}}$.

Sum, $\frac{2a^2-2b^2}{a^2+b^2}$

14. Add $\frac{\sqrt{7}}{\sqrt{7}+\sqrt{3}}$ and $\frac{\sqrt{3}}{\sqrt{7}-\sqrt{3}}$ together.

Sum, $\frac{5}{2}$

15. Divide $18+26\sqrt{-1}$ by $3+\sqrt{-1}$.

Quot. $8+6\sqrt{-1}$.

16. Required the square root of $13-20\sqrt{-3}$?

Ans. $5-2\sqrt{-3}$.

Imaginary surds are of great importance, in the investigation of several valuable formula, in the arithmetic of sines.

Section XI.

EQUATIONS IN GENERAL.

114. It has been remarked, page 77, that quadratic equations sometimes admit of more answers than one; the principles on which the ambiguity of quadratic equations depends, are productive of similar results in equations of the higher orders.

In the general equation $x^n - px^{n-1} + qx^{n-2} \ldots\ldots \pm r = 0$, where p, q, r, are supposed to be given, the different values, of which x is susceptible, are called the roots of the equation.

115. If two binomials, as $x-a$, $x-b$, be multiplied together, the product $x^2 - ax - bx + ab$, or $x^2 - (a+b)x + ab$, is manifestly a quadratic; if now, $x-a=0$, or $x-b=0$, we have evidently $x^2 - (a+b)x + ab = 0$. On the other hand, the given quadratic equation $x^2 - px + q = 0$, may be resolved by making $a+b=p$, $ab=q$, and determining a and b. For, on these assumptions being made, the equations become identical, and their conditions are fulfilled by taking $x=a$, or $x=b$; the equation $x^2 - px + q = 0$, has therefore, two positive roots, which are both possible when $\frac{1}{4}p^2$ is greater than q, and impossible when q exceeds $\frac{1}{4}p^2$. This expression corresponds to form 3d, *A*, (art. 62.)

116. Again, $\overline{x-a} \times \overline{x+b} = x^2 - ax + bx - ab$, which being supposed $=0$, corresponds to $x^2 + px - q = 0$, or to $x^2 - px - q = 0$, according as b is greater or less than a.

The conditions of this equation are answered by taking $x-a=0$, or $x+b=0$; the equation has, therefore, two roots, a, and $-b$. These expressions correspond to forms 1, 2, *A*, (art. 62.) Those equations consequently admit of a negative and a positive root, which are both possible, because when the square is completed, the second member consists of positive quantities.

117. Moreover, $(x+a).(x+b) = x^2 + ax + bx + ab$, which being supposed $=0$, agrees with $x^2 + px + q = 0$.

Hence an equation of this form is resolved by making $a+b=p$, $ab=q$, and determining a and b. The roots of this equation are manifestly both negative; and as in (art. 115,) both possible or both impossible.

118. It is worthy of remark, that in the equation $x^2 - px + q = 0$, the signs are alternately $+$ and $-$, that is, they are *twice* changed; and the equation has *two* positive roots. But in the equations (art. 116,) the same sign is *once* continued, and the sign *once* changed; and

these equations have *one* affirmative and *one* negative root. In the equation (art. 117,) the sign is *twice* continued; and this equation has *two* negative roots.

It also appears, from what is above shown, that every quadratic equation has two roots, which are both possible, or both impossible.*

119. Assuming three factors, $(x-a).(x-b).(x-c)=x^3-(a+b+c)x^2+(ab+ac+bc)x-abc$, which being supposed $=0$, will be identical with $x^3-px^2+qx-r=0$, if $p=a+b+c$, $q=ab+ac+bc$, and $r=abc$. But the conditions of the former are answered by making $x-a=0$, $x-b=0$, or $x-c=0$; therefore, the latter has three positive roots, a, b, and c. It has likewise *three* changes of the signs.

120. If instead of $x-c$ we take $x+c$, our equation $(x-a).(x-b).(x+c)=x^3-(a+b-c)x^2+(ab-ac-bc)x+abc=0$, will, manifestly, have *two positive* roots, a and b, and one negative root, $-c$. In this case, if $a+b>c$, the sign of the second term is $-$, the equation may, therefore, be expressed $x^3-px^2\pm qx+r=0$, in which there are *two* changes of the signs, and one continuation of the same sign. If $c>a+b$, the second term will have the sign $+$, but the third $-$, because in that case $(a+b).c>(a+b)^2$, and therefore, $>ab$. The equation then becomes $x^3+px^2-qx+r=0$; having, as before, *two* changes of the signs, and *one* continuation of the same sign.

121. A cubic equation, composed of the factors $(x-a).(x+b).(x+c)=x^3+(b+c-a)x^2-(ab-ac+bc)x-abc=0$ has plainly one positive root, a, and two negative roots, $-b$, and $-c$. But in this case, if $b+c>a$, the sign of the second term is $+$; and the equation may be expressed $x^3+px^2\pm qx-r=0$, having *one* change of the signs, and

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* In form 3, A, (art. 62,) when $b=a^2$, the roots are equal to each other. In forms 1 and 2, if $2a=0$, the equation becomes a simple quadratic, and the roots are equal quantities with contrary signs. Every simple quadratic has, therefore, a positive and negative root.

two continuations of the same sign. If $a>b+c$, the second and third terms are both negative, because $(b+c)a>(b+c)^2>bc$; the equation, therefore, may be expressed $x^3-px^2-qx-r=0$, having, as before, *one* change of the signs, and *two* continuations of the same sign.

122. If we use the factors $x+a, x+b, x+c$, their product $x^3+(a+b+c)x^2+(ab+ac+bc)x+abc$, being put $=0$, the equation may be expressed, $x^3+px^2+qx+r=0$, in which there are *three* continuations of the same sign. The equation has likewise *three* negative roots, $-a$, $-b$, and $-c$.

By pursuing this inquiry it will be found, that any equation of this kind, admits of as many roots as there are units in the index of the highest power of the unknown quantity; that the number of positive and negative roots, will be, respectively, equal to the number of changes in the signs, and the number of continuations of the same sign. It likewise appears, that the last term, or absolute number, is the continued product of all the roots with their signs changed.

123. It may be observed, that as every cubic equation is composed of three factors, and every quadratic of two; a cubic equation may always be considered as the product of a simple and a quadratic equation. But (art. 118,) every quadratic has, either two possible, or two impossible roots; hence, a cubic equation, the terms of which are possible, having *one* impossible root, has *two;* and, as in the multiplication of compound quantities, containing impossible parts, those impossible parts can disappear only when two like roots are multiplied together, it follows, that every cubic equation, consisting of possible quantities, has, at least, one possible root.

124. In like manner, it appears that every equation consisting of possible quantities, having an odd number of roots, has, at least, one of those roots possible. And that every equation which is made up of possible quantities, has all its roots possible, or an *even* number of impossible roots.

125. When the roots of an equation are integral, they may sometimes, be found with great facility, by seeking the divisors of the last term, and substituting them in place of the unknown quantity, till one or more be found which answer the conditions of the equation. (See art. 122.)

When one root has been found, the equation may be depressed, by connecting that root, with its sign changed with the unknown quantity, and dividing the given equation by the sum.

EXAMPLES.

1. Given $x^3-3x^2+5x-15=0$, to find the value of x.

(Art. 119,) the roots if possible, are all positive.*

Also, the divisors of 15 are 1, 3, 5, 15.

Now, by substituting these for x,

$1-3+5-15=-12$. $\therefore$ 1 is not a root.

$27-27+15-15=0$. $\therefore$ 3 is a root.

$x-3=0$, and $\frac{x^3-3x^2+5x-15}{x-3}=x^2+5=0$.

Whence, $x=\pm\sqrt{-5}$.

$\therefore$ the roots are 3, $+\sqrt{-5}$, and $-\sqrt{-5}$.

2. Given $x^3-2x^2-5x+6=0$, to find x.

Results, 1, 3, -2.

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* The rules, in the foregoing articles, for determining the signs of the roots from the changes in the signs of the terms composing the equation, being founded on the supposition, that each root has but *one* sign, do not apply to impossible roots; because the negative signs under the radicals, when developed by multiplication, are combined with those of the roots, and therefore, change the signs of the terms of which the equation is composed.

3. Given $x^3+6x^2-7x-60=0$, to find x.

Results, 3, -4, -5.

4. Given $x^3+3x^2-6x-8=0$, to find x.

Results, 2, -1, -4.

5. Given $x^3-2x+4=0$, to find x.

Results, -2, $1+\sqrt{-1}$, $1-\sqrt{-1}$.

6. Given $x^4-10x^3+35x^2-50x+24=0$, to find x.

Results, 1, 2, 3, 4.

7. Given $x^4-8x^3+23x^2-64x+120=0$, to find x.

Results, 5, 3, $2\sqrt{-2}$, $-2\sqrt{-2}$.

126. When the roots are not integral, they may generally be determined by approximation.* For this purpose, various rules have been investigated. Among these the following is probably the most convenient in practice. The demonstration is given in the subsequent article.

To find the root of a general equation.

1. If all the terms of the equation are not on one side, by transposition, place them so; and arrange them according to the powers of the unknown quantity, placing the highest power on the left hand. If any of the lower powers are not contained in the equation, consider each one omitted, as having a *cipher* for its *co-efficient*.

2. Place the co-efficients and the absolute number with their proper signs, in order, in a horizontal line.

* If all the roots of the equation are impossible, this method is not applicable, (see art. 106,) but possible roots may always be approximated.

3. Find by trial the first root figure, attending to its value as being units, tens, tenths, or hundreths, &c. and place it to the right of the absolute number.

4. Multiply the first co-efficient by the root figure, and add the product to the second co-efficient; multiply the sum by the root figure, and add the product to the third co-efficient; proceed thus to the end of the line, adding the last product to the absolute number. Again, multiply the first co-efficient by the root figure, and add the product to the sum under the second co-efficient; multiply the resulting sum by the root figure, and add the product to the sum under the third co-efficient; and so on, stopping under the last co-efficient. Repeat the process, stopping each succeeding time, one term nearer to the left hand, till the last sum falls under the second co-efficient.

5. Try how often the last sum under the last co-efficient is contained in the sum under the absolute number, and take the result for the next root figure.

6. Using the first co-efficient, and the last sum in each column, instead of the co-efficients and absolute number, proceed with this new root figure as with the preceding one.

7. Obtain another root figure in the manner last mentioned, and thus continue the operations as far as necessary.

Note 1.—In multiplying by each root figure, attention must be given to its value. Thus, if it is of the order of tens, the multiplication must be made by the number of tens which it represents; and so, for other values. Also, in the multiplications and additions, regard must be had to the signs of the numbers.

2. The signs of the successive sums, under the absolute number, must continue throughout the operation, the same as that of the absolute number. If the opera-

tion for any of the root figures cause the sign to change, a less value must be taken for that figure. This will not unfrequently occur with regard to the second root figure, but it will seldom be the case for the others.

3. After two or three root figures have been obtained, and the multiplications and additions corresponding to them have been completed, the succeeding parts of the operation may be contracted in the following manner. Cut off the right hand figure of the sum, in the column under the last co-efficient; the two right hand figures of the sum in the preceding column; the three right hand figures of the sum in the column preceding that, and so on.

If either of the figures next to the right of the marks of separation is 5, or more than 5, add, mentally, a unit to the first figure on the left of the mark, when using it in the succeeding multiplication and addition. Repeat the same contraction for each of the following root figures.

These contractions may commence, in cubic equations, after the second or third decimal figure in the root is obtained; in biquadratic equations, after the first or second decimal figure; and in higher equations, after the first decimal figure. And if the operation is closed when the sum under the absolute number is reduced to two figures, all the figures in the root will be true.*

EXAMPLES.

1. Given $3x^4-4x^3+2x-1000=0$, to find the value of x. Ans. 4.342447603.

* This rule, improved from Young's Algebra, was communicated by my friend John Gummere, of Burlington.

3	0	—4	+2	—1000(4.342447603
	12	48	176	712
	12	44	178	—288
	12	96	560	248.2803
	24	140	738	—39.7197
	12	144	89.601	37.39678208
	36	284	827.601	—2.32291792
	12	14.67	94.083	1.89781666
	48.9	298.67	921.684	—.42510126
	.9	14.94	13.235552	37988512
	49.8	313.61	934.919552	—4521614
	.9	15.21	13.318496	3799436
	50.7	328.82	948.23804,8	—722178
	9	2.0688	.67028	664909
	51.6	330.8888	948.90833	—57269
	.12	2.0736	.67048	56994
	51.72	332.9624	949.5788,1	—275
	12	2.0784	.1340	285
	51.84	335.04,08	949.7128	
	12	10	1340	
	51.96	335.14	949.846,8	
	.12	10	12	
	5,2.08	335.24	949.859	
		10	12	
		3,35;34	94,9,8,7,1	

2. Given $x^5+2x^4+3x^3+4x^2+5x-54321=0$, to find the value of x. Ans. 8.41445475.

1	2	3	4	5	—54321(8.41445475
	8	80	664	5344	42792
	10	83	668	5349	—11529
	8	144	1816	19872	11088.97344
	18	227	2484	25221	—440.02656
	8	208	3480	2501.4336	304.11052
	26	435	5964	27722.4336	—135.91604
	8	272	289.584	2620.0064	122.02904
	34	707	6253.584	30342.440,0	—13.88700
	8	16.96	296.432	68.612	12.21504
	42.4	723.96	6550.016	30411.052	—1.67196
	.4	17.12	303.344	68.690	1.52700
	42.8	741.08	6853.3,60	30479.74,2	—14496
	.4	17.28	7.8	27.52	12216
	43.2	758.36	6861.2	30507.26	—2280
	.4	17.44	7.8	27.52	2135
	43.6	77,5.80	6869.0	30534.7,8	—145
	.4		7.8	2.8	153
	44.0		6,87,6.8	30537.6	
				2.8	
				30,5,4,0.4	

3. Given $x^3+9x^2+4x=80$, to find x.

Result, $x=2.4721859$.

4. Given $x^3+x^2+x=90$, required the value of x.

Ans. $x=4.1028323$.

5. Given $x^3+10x^2+5x=2600$, required the value of x.

Ans. 11.00679934.

6. Given $2x^4-16x^3+40x^2+30x=110$, required the value of x. Ans. $x=5.0569081$.

127. To show the rationale of the process directed in the last article, I begin with the cubic equation

$$ax^3+bx^2+cx+d=0.$$

Let the first figure in the root be indicated by r, regard being paid to its *local value*, and the remaining part of the root by y; then $x=y+r$.

$$\left.\begin{array}{rrrr} \text{Hence, } ax^3= & ay^3+3ary^2+3ar^2y+ar^3 & \\ bx^2= & by^2+2bry+br^2 & \\ cx= & cy+cr & \\ d= & d & \end{array}\right\}=0.$$

Collecting the co-efficients of like powers of y,

$$ay^3+b'y^2+c'y+d'=0.$$

It is obvious that $d'=\{(ar+b)r+c\}r+d$;

$$c'=(ar+b)r+(ar+b)r+ar+c;\ b'=ar+ar+ar+b.$$

But these are the quantities found by the fourth precept.

Again, since $r>y$, $c'y+d'$ approximates to 0, or $y=\frac{-d'}{c'}$ nearly; but this is the mode prescribed in the fifth precept for finding the next figure of the root.

Denoting the number obtained by the last operation by s, and the remaining part of y by z, so that $y=z+s$, we shall obtain a new equation $az^3+b''z^2+c''z+d''=0$, in which $d''=\{(as+b')s+c'\}s+d'$; $c''=(as+b')s+(as+b')s+as+c'$; $b''=as+as+as$; whence an approximate value of z, or a new figure of the root is manifestly deducible from this new equation, as before.

Assuming now the general equation

$$ax^n+bx^{n-1}\ldots\ldots\ mx^2+nx+p=0.$$

And denoting as before, the first figure of the root by r, and making $x=y+r$, we have

$$\left.\begin{array}{lllll} ax^n=ay^n+nary^{n-1}\ldots & \ldots.\&c. & nar^{n-1}y+ar^n \\ bx^{n-1}= & by^{n-1}\ldots & \&c. & \overline{n-1}.br^{n-2}y+br^n \\ \&c. & \ldots & \&c. & \\ mx^2= & & & 2mry+mr^2 \\ nx= & & & ny+nr \\ p= & & & p \end{array}\right\}=0.$$

Or $ay^n+b'y^{n-1}+\ldots$ &c. $n'y+p'=0$, in which the quantities b', n', p', are composed of the co-efficients a, b, &c. and the powers of r combined, as directed in the fourth precept. And here as before, $y=-\frac{p'}{n'}$ nearly.

Hence it is obvious, that the successive figures of the root may be obtained by the same kind of process; whatever may be the index of the highest power of the unknown quantity.

INDETERMINATE PROBLEMS.

128. When a problem is given, in which the number of unknown quantities employed, is greater than the number of independent equations furnished by the conditions of the problem; one, at least, of those quantities, may be assumed at pleasure. Such problems, therefore, generally admit of an indefinite number of answers. There are, however, certain conditions sometimes annexed, by which the number of answers is partially limited. For example, the answers are required to be whole positive numbers, or they are required to be square or cube numbers. Such problems are termed indeterminate, or unlimited, though in some instances, each unknown quantity admits of but one value. If simple powers only of the unknown quantities are included in

the equations, the problem is said to be of the first degree.

Case 1.

*To find the values of x and y, in whole positive numbers, from the equation $ax=by+c$; a, b, c, being given numbers, positive or negative.**

Here $x=\frac{by+c}{a}$, and as x is to be a whole number, $\frac{by+c}{a}$ must also be a whole number.

Now, if this quantity be multiplied by a whole number, the product must, evidently, be a whole number; also the sum or difference of this quantity, or either of its multiples, and any whole number, must necessarily be a whole number. Let, therefore, $\frac{by+c}{a}$ be thus changed till we obtain $\frac{y+c'}{a}=wh$,† which put $=p$;† then $y=ap-c'$, a quantity that must be a whole number, because a, p, and c' are whole numbers. And this value being substituted for y in the equation $x=\frac{by+c}{a}$, the value of x will be obtained in terms of p, and given numbers.

Assuming then $p=0$, 1, 2, 3, &c. successively, (omiting such numbers as make x or y negative,) the various numerical values of x and y become known.

The number of answers will be limited when the signs of p in the values of x and y are unlike; but unlimited when p has the same sign in both.

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* If a and b have a common divisor, it must also be a divisor of c, or the problem is impossible.

† This expression is used to designate any *whole number*.

EXAMPLES.

1. Given $19x=14y+15$, to find the values of x and y in whole positive numbers.

$$x=\frac{14y+15}{19}=wh. \quad \therefore \frac{14y+15}{19}\times 4=$$

$$\frac{56y+60}{19}=2y+3+\frac{18y+3}{19}=wh.$$

But $\frac{19y}{19}=wh.$ $\therefore \frac{19y}{19}-\frac{18y+3}{19}=\frac{y-3}{19}=wh=p.$

$\therefore y=19p+3$, and $x=\frac{\overline{19p+3}\times 14+15}{19}=14p+3.$

If now, we assume p successively $=0, 1, 2\ 3$, &c.

$y=3, 22, 41, 60$, &c.

$x=3, 17, 31, 45$, &c.

Here the number of answers is evidently unlimited.

2. Given $11x+17y=987$, to find x and y in whole positive numbers.

$$x=\frac{987-17y}{11}=89-y+\frac{8-6y}{11}=wh. \quad \therefore \frac{8-6y}{11}=wh,$$

$$\frac{8-6y}{11}\times 2=\frac{16-12y}{11}=1-y+\frac{5-y}{11}=1-y-\frac{y-5}{11}=wh.$$

$$\therefore \frac{y-5}{11}=wh=p.$$

Whence $y=11p+5$,

And $x=\frac{987-(11p+5)\times 17}{11}=82-17p.$

Assuming $p=0, 1, 2, 3, 4,$

$$y=5, 16, 27, 38, 49.$$

$$x=82, 65, 48, 31, 14.$$

Which are all the possible values in whole positive numbers.

3. Given $7x+9y=2342$, to find the number of values of x and y in whole positive numbers.

$$x=\frac{2342-9y}{7}=334-y-\frac{2y-4}{7}=wh. \quad \therefore \frac{2y-4}{7}=wh,$$

$$\frac{2y-4}{7}\times 4=\frac{8y-16}{7}=y-2+\frac{y-2}{7}=wh. \quad \therefore \frac{y-2}{7}=wh=p.$$

$$\therefore y=7p+2, \text{ and } x=332-9p.$$

From the first of these expressions we perceive, that the least value of p is 0, and from the second, that the greatest value of p is 36. Hence the required number is 37.

3. Given $5x+7y+9z=337$, to find the number of values of x, y, and z, in whole positive numbers.

$$x=\frac{337-7y-9z}{5}=67-y-z+\frac{2-2y-4z}{5}=wh.$$

$$\therefore \frac{2-2y-4z}{5}=wh,$$

$$\text{And } \frac{2-2y-4z}{5}\times 2=\frac{4-4y-8z}{5}=wh.$$

$$\text{But } \frac{5y-5+10z}{5}=wh.$$

$$\therefore \frac{5y-5+10z}{5}+\frac{4-4y-8z}{5}=\frac{y-1+2z}{5}=wh=p.$$

$$\therefore y=5p-2z+1, \; x=66-7p+z.$$

Assuming now z successively equal to 1, 2, 3, &c. we shall have the corresponding values of x and y, the greatest and least values of p, and the number of answers as follows:

z	x	y	p Gr.	Least.	No. of Ans.
1	$67-7p$	$5p-1$	9	1	9
2	$68-7p$	$5p-3$	9	1	9
3	$69-7p$	$5p-5$	9	2	8
4	$70-7p$	$5p-7$	9	2	8
5	$71-7p$	$5p-9$	10	2	9
6	$72-7p$	$5p-11$	10	3	8
7	$73-7p$	$5p-13$	10	3	8
8	$74-7p$	$5p-15$	10	4	7
9	$75-7p$	$5p-17$	10	4	7
10	$76-7p$	$5p-19$	10	4	7
11	$77-7p$	$5p-21$	10	5	6
&c.	&c.	&c.	&c.	&c.	&c.

Here we may observe, there are seven successive values of z, (beginning with the 5th,) which produce no change in the *greatest value* of p; and there are alternately, two and three equal *least values* of p; and this order will evidently continue as long as successive values of z be assumed.

The last three columns of the above table may, therefore, be continued without the former ones.

Gr. val. of p,	9,	9,	9,	9,	10,	10,	10,	10,	10,	10,	10,	11,	11,	11.
Least do.	1	1	2	2	2	3	3	4	4	4	5	5	6	6
No. of Ans.	9	9	8	8	9	8	8	7	7	7	6	7	6	6

Gr. val. of p,	11	11	11	11	12	12	12	12	12	12	12	13	13	13
Least do.	6	7	7	8	8	8	9	9	10	10	10	11	11	12
No. of Ans.	6	5	5	4	5	5	4	4	3	3	3	3	3	2

Gr. val. of p	13	13	13	13	14	14	14	14
Least,	12	12	13	13	14	14	14	15
No. of Ans.	2	2	1	1	1	1	1	0

The number of answers, or sum of the numbers in the last column, is 169.

4. Given $14x=5y+19$, to find the least possible values of x and y, in whole positive numbers.

Ans. $x=6$, $y=13$.

5. Given $11x+5y=254$, to find all the values of x and y in whole positive numbers.

Ans. $x=19, 14, 9, 4.$
$y=9, 20, 31, 42.$

6. Given $9x+13y=2000$, required the number of values of x and y? Ans. 17.

7. Required to divide 100 into two parts, so that one of them may be divisible by 7, and the other by 11?

The parts are 56 and 44.

8. Given $17x+19y+21z=400$, required the number of values of x, y, and z. Ans. 10.

9. Required to pay 1000 dollars, in French crowns, and five franc pieces, so that the number of coins used shall be the least possible, what number of each kind will be necessary, and how many ways can that sum be paid in those coins, the French crown being 110 cents, and the five franc pieces 93 cents?

Ans. 833 crowns, 90 five franc pieces, and 9 different ways.

10. How many gallons of liquor at 12 cents, 15 cents, and 18 cents per gallon, may be mixed, to compose 300 gallons at 17 cents per gallon?*

Ans. { at 12 cents, 1, 2, 3, &c. to 49.
at 15 cents, 98, 96, 94, &c. to 2.
at 18 cents, 201, 202, &c. to 249.

11. In how many ways can £1053 sterling be paid

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* When more equations than one are given, one, at least, of the unknown quantities may be eliminated, and the equations reduced to one.

without using any coins besides guineas and moidores; the guinea being 21*s.* sterling, and the moidore 27?

Ans. 112 ways.

12. A foreigner having a bill of $1 75, to pay at a hotel, offers napoleans in payment, the landlord agrees to receive them on condition that Turkish sequins shall be taken as change; how many pieces must be used, a napolean being worth $7 25, and a sequin rated at $2 10. Ans. 35 napoleans, and 120 sequins.

13. Given $7x+9y+23z=9999$; of how many values will x, y, and z admit? Ans. 34365.

Case 2.

129. *To find a whole number which being divided by any given numbers, shall leave given remainders.*

Denoting the required number by x, the given divisors by a, b, c, &c. and the remainders by f, g, h, &c. we shall have $\frac{x-f}{a}$, $\frac{x-g}{b}$, $\frac{x-h}{c}$, &c. severally equal to whole numbers.

Put $\frac{x-f}{a}=p$, find the value of x, and substitute it for x in the second fraction; reduce this new fraction, as in the former case, so that the co-efficient of p may be a unit, and put the fraction thus obtained $=q$; find the value of p, and thence of x, in terms of q and given numbers. Substitute the value of x in the third equation, and so proceed. The last value of x will be the number sought.

EXAMPLES.

1. Required a number which being divided by 7, 9, 10, and 11, shall leave the remainders 5, 4, 7, and 9, respectively.

Let x= the number sought.

Then $\frac{x-5}{7}, \frac{x-4}{9}, \frac{x-7}{10}, \frac{x-9}{11}$, are whole numbers.

$$\frac{x-5}{7}=wh=p. \quad \therefore x=7p+5.$$

Whence, $\frac{7p+5-4}{9}=\frac{7p+1}{9}=wh.$

$$\therefore \frac{7p+1}{9}\times 4=\frac{28p+4}{9}=3p+\frac{p+4}{9}=wh.$$

$$\therefore \frac{p+4}{9}=q, \text{ and } p=9q-4. \quad \therefore x=63q-23.$$

$$\therefore \frac{63q-30}{10}=6q-3+\frac{3q}{10}=wh. \quad \therefore \frac{3q}{10}=wh.$$

And consequently,

$$\frac{q}{10}*=wh=r. \quad \therefore q=10r, \text{ and } x=630r-23.$$

$$\therefore \frac{630r-32}{11}=57r-3+\frac{3r+1}{11}=wh. \quad \therefore \frac{3r+1}{11}=wh.$$

And $\frac{3r+1}{11}\times 4=\frac{12r+4}{11}=r+1+\frac{r-7}{11}=wh.$

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* If a fractional expression $\frac{a}{b}=wh$, and we divide the terms into their primes, it is plain that all the prime numbers contained in the denominator, must be also contained in the numerator; thus, $\frac{3q}{10}=\frac{3.q}{2.5}=wh$, where the primes 2 and 5 not being contained in the 3, must be contained in q, and therefore, $\frac{q}{10}=wh$.

$$\therefore \frac{r-7}{11}=wh=s. \quad \therefore r=11s+7.$$

And $x=6930s+4387=$(if $s=0$) 4387.

Assuming $s=1$, 2, &c. other values would arise.

2. Required a number which being divided by 2, 3, 4, 6, shall leave no remainder; but being divided by 7, the remainder shall be 6?

The least common multiple of 2, 3, 4, 6,* is 12.

Let, therefore, $12x=$ the number sought.

$$\therefore \frac{12x-6}{7}=wh. \quad \therefore \frac{12x-6}{7}\times 3=\frac{36x-18}{7}=5x-2+$$

$$\frac{x-4}{7}=wh\text{; and } \frac{x-4}{7}=wh=p. \quad \therefore x=7p+4.$$

And $12x=84p+48=48$, 132, 216, &c.

3. Required the least whole number, which being divided by 3, 5, 7, and 8, shall leave the remainders 1, 3, 5, and 0, respectively? Ans. 208.

4. Required the least whole number, which being divided by 9, 10, 11, and 12, shall leave the remainders 4, 5, 6, and 7, respectively? Ans. 1975.

5. What is the least whole number, which being divided by each of the nine digits shall leave no remainder; but divided by 17 the remainder shall be 13?

Ans. 40320.

6. In what year of the Christian era, was the solar cycle 15, the lunar cycle 3, and the Roman indiction 14?† Ans. 1826.

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* See Arithmetical Expositor, Part I, page 73.

† The solar cycle is a period of 28 years, at the end of which, in case a centurial year has not intervened, the days of the week always return to the same days of the month. To the year of

7. Required the least whole number, which being divided by 2, 3, 4, and 5, shall leave a remainder of 1; but divided by 7, there shall nothing remain.

Ans. 301.

MISCELLANEOUS EXAMPLES.

130.—1. A bankrupt owes A. twice as much as he owes B., and he owes to C. as much as to A. and B. together; the sum to be divided is 600 dollars; how much must each receive?

Ans. A. 200, B. 100, C. 300.

2. A. can perform a piece of work in 7 days, and B. in 9 days; in what time would they jointly effect it?

Ans. $3\frac{15}{16}$ days.

3. A labourer being employed on condition, that for every day he worked he should receive 50 cents, and for every day he was idle, he should forfeit 20 cents, finds at the end of 500 days, only 89 dollars due; how many days did he work? Ans. 290.

4. There are two numbers in the ratio of 4 to 5, and the sum of their squares is 1476; what are the numbers?

Ans. 24 and 30.

5. A Greek epitaph, designed for the tomb of Diophantus, is said to have stated that he passed one sixth of his life in childhood; one twelfth in adolescence; that after one seventh and five years more had been passed

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the Christian era, add 9, and divide by 28, the remainder will be the number of the cycle.

The lunar cycle is a period of 19 years, at the expiration of which, the new and full moons return nearly to the same time of the year. To the year of the Christian era add 1, and divide by 19, the remainder is the lunar cycle.

The Roman indiction is not an astronomic period, it consists of 15 years. To a given year add 3, and divide by 15, the remainder is the number of the indiction.

If, in either of these cases, no remainder occurs, the divisor must be taken as the number of the cycle.

in a married state, he had a son, who lived to half his own age, and whom he survived four years; what then was the age of Diophantus? Ans. 84 years.

6. A person's age in years is a number, consisting of two digits; $\frac{1}{7}$ of this number is a mean proportional between these digits; and two years hence his age will be a third proportional to those digits, beginning with the tens; what is the age? Ans. 14 years.

7. Given $x+y+z=9$, $xy+xz+yz=26$, $xyz=24$, to find the values of x, y, and z.

Result, $x=2$, 3, or 4.

8. By selling a piece of muslin at a certain price per yard, I gained the prime cost of 9 yards, which was just as much per cent. as the number of yards in the piece; what was that number? Ans. 30 yards.

9. There are 4 numbers in continued proportion, the sum of the first and last 1728, and the sum of the other two 1152; what are the numbers?

Ans. 192, 384, 768, and 1536.

10. The hypothenuse of a right angled triangle is x^{3x}, and the other sides x^{2x}, and x^{x}; what is the area?

Ans. $\frac{1}{4}\sqrt{\{8+6\sqrt{5}.\}}$

11. There is a number consisting of three digits in arithmetical progression, which number being divided by the sum of the digits the quotient will be 48, but if from the number 198 be subtracted, the remainder will be expressed by the same digits in an inverted order. Quere the number? Ans. 432.

12. The sum of the squares of two numbers being multiplied by the quotient arising from the division of the less by the greater, produces 83.2, and the difference of the squares multiplied by the quotient of the greater divided by the less, produces 1920; what are the numbers? Ans. 20 and 4.

13. A ball falling from the top of a tower, is observed

to descend one fourth of the distance in the last second of the time; required the height of the tower, heavy bodies being known to fall $16\frac{1}{12}$ feet during the first second, and to describe spaces, which, reckoned from the beginning of the fall, are as the squares of the times?

Ans. $(28\pm16\sqrt{3})16\frac{1}{12}$ feet.

14. What are the values of x and z, from the equations $x^3z+xz^3=546560$, $x^4+z^4=1086992$?

Ans. $x=32$, $z=14$.

15. Given $x+y+z+v=56$, $x^2+y^2+z^2+v^2=910$, $xv+2y^2-z^2=6$, and $z=2y$, to find the values of x, y, z, and v. Result, $x=8$, $y=9$, $z=18$, $v=21$.

16. Required to find a number, which being any way divided into two unequal parts, the greater part added to the square of the less, shall be equal to the less part added to the square of the greater? Ans. 1.

17. Given $xy=125x+300y$, and $y^2-x^2=90000$, to find x and y by a quadratic.

Result, $x=400$, $y=500$.

18. Given (x^3+1), (x^2+1), $(x+1)=30x^3$, to find the value of x, by a quadratic equation.

Result, $x=\frac{1}{2}(3\pm\sqrt{5}.)$

19. The sum of three numbers in harmonical proportion is 26, and their continued product 576; required the numbers by a quadratic equation?

Ans. 12, 8, and 6.

20. The sum of two numbers is 152, and the cube root of the square of their difference multiplied by the square root of the cube of the same difference, produces 8192; what are the numbers? Ans. 44 and 108.

21. The sum of two numbers added to the sum of their squares is 120, and the product of the same numbers 45; what are they? Ans. 9 and 5.

22. Given $x^3+xy^2=4640y$, and $x^2y-y^3=537.6x$, what are the numerical values of x and y?

Ans. $x=40$, $y=16$.

23. Given $x+y=z$, $z^3-x^3+y^6=470$, $z^3-x^3-y^3=468$, to find x, y, and z.

Result, $x=12$, $y=1$, $z=13$.

24. Given $v^2+vx+vy+vz=252$, $x^2+xv+xy+xz=504$, $y^2+yv+yx+yz=396$, $z^2+zv+zy+zx=144$, to find v, x, y, and z.

Result, $v=7$, $x=14$, $y=11$, $z=4$.

25. Given $x^4-2x^3+x=a$, to find x by a quadratic equation. Result, $x=\frac{1}{2}+\sqrt{(\frac{3}{4}+\sqrt{a+\frac{1}{4}})}$.

26. Given $x^2+x(y+z)=a$, $y^2+y(x+z)=b$, $z^2+z(x+y)=c$, to find x, y, and z.

Results, $\begin{cases} x=\dfrac{a}{\sqrt{(a+b+c)}} \\ y=\dfrac{b}{\sqrt{(a+b+c)}} \\ z=\dfrac{c}{\sqrt{(a+b+c)}} \end{cases}$

27. Given $2x^3-x^4-x^2+2x^3y-y^2=2xy=\sqrt{x^5y}-\sqrt{x^3y}$, to find x and y. Result, $x=5$, $y=20$.

28. Required the roots of the equation $4x^4+8x^3-89x^2+28x+49=0$, by quadratics only?

Ans. $1, \frac{7}{2}, \dfrac{-13+\sqrt{113}}{4}, \dfrac{-13-\sqrt{113}}{4}$

29. There are three numbers in geometrical progression, whose continued product is 4096, and the sum of the extremes is 68; what are the numbers?

Ans. 4, 16, and 64.

30. There are two numbers expressed by the same two digits, and the difference of their squares is 1485; quere the numbers? Ans. 14 and 41.

31. What two numbers are those whose product, difference of their squares, and quotient of their cubes are all equal? Ans. $\frac{3}{4}+\frac{1}{4}\sqrt{5}$, and $\frac{1}{2}+\frac{1}{2}\sqrt{5}$.

32. The product of two numbers is 10, and the product of their sum by the sum of their squares is 203; required the numbers found by a quadratic equation?

Ans. 5 and 2.

33. There are three numbers in geometrical progression, the difference of whose differences is 6, and the sum of the numbers 42; what are the numbers?

Ans. 24, 12, and 6.

34. There are three numbers in harmonical proportions, the difference of whose differences is 2, and the product of the extremes 72; what are the numbers?

Ans. 6, 8, and 12.

35. Required three equidifferent numbers, such that if the first be increased by 1, the second by 2, and the third by the first, the sums may constitute an harmonical progression; but if 3 be added to the second, the sum may be a mean proportional between the sum of the numbers and the first diminished by $\frac{1}{2}$?

Ans. 5, 6, and 7.

36. Required two such squares, that their difference shall be to the square root of the less as 3 to 7, and the square roots of the numbers to each other as 5 to 2? Ans. $(\frac{10}{21})^2$, and $(\frac{4}{21})^2$.

37. Given the sum of the cubes of two numbers $=35$, and the sum of their 9th powers $=20195$; what are the numbers? Ans. 3 and 2.

38. Given $9\frac{x^2}{y^2}+36\frac{x}{y}=85$, $\frac{3xy^2}{5}+\frac{51xy^2}{10}=\frac{102x}{5}+16$, required the values of x and y?

Ans. $x=3\frac{1}{3}$, $y=2$.

39. What two numbers are those, whose difference is 4, and their product multiplied by the sum of their squares 480? Ans. 6 and 2.

40. Given $x^2y\sqrt{xy}=a$, $xz^2\sqrt{xz}=b$, $y^2z\sqrt{zy}=c$, to find the values of x, y, and z.

Ans. $x=(\frac{a^4p^9}{c^6})^{\frac{1}{19}}$, $y=(\frac{c^4p^9}{b^6})^{\frac{1}{19}}$

$z=(\frac{b^4p^9}{a^6})^{\frac{1}{19}}$, p being $=\sqrt[4]{abc}$.

41. Required the values of x, y, and z, from the equations $x(x+y+z)=a$, $y(x+y+z)=b$, $z(x+y+z)=c$.

Ans. $x=\sqrt{\frac{bc}{a}}$, $y=\sqrt{\frac{ac}{b}}$, $z=\sqrt{\frac{ab}{c}}$

42. There are three numbers in harmonical proporportion, the sum of the first and third is 18, and their continued product 576, what are the numbers?

Ans. 6, 8, and 12.

43. In how many different ways is it possible to pay £1000, without using any other coins than crowns, guineas, and moidores; a crown being 5*s*., a guinea 21*s*., and a moidore 27*s*.? Ans. 70734.

44. Given $x^2+y^2+z^2=266.5$, $x^2+y+z^2=176.5$, $(x+y+z)y=280$. Required the values of x, y, and z. Ans. $x=10.5$, $y=10$, $z=7.5$

45. Required two cube numbers, such that the first multiplied into the product of their roots, shall be equal to the second; but the second multiplied into the product of their roots, shall be equal to 64 times the first?

Ans. 8 and 64.

46. There are three numbers in geometrical progression, such that if the third be diminished by the first, the first, second, and the remainder, shall be in arithmetical progression; but the first and third being each increased by the second, and the second increased by 2,

the numbers will be in harmonical progression; what are the numbers? Ans. 1, 2, 4.

47. Given $x^2+y^2-x-y=249740$, $xy+x+y=8516$, to determine the values of x and y.

Result, $x=500$, $y=16$.

48. Given $x^4+y^4+x^2+y^2=238632-2x^2y^2$,

$y^4+z^4+z^2+y^2+2y^2z^2=1640$,

$x^2+y^2+z^2=\sqrt{(275100-x^2-y^2-z^2)}$

to find the values of x, y, and z.

Ans. $x=22$, $y=2$, $z=6$.

49. The sum of five numbers in geometrical progression is 242, and the fourth difference is 2; required the numbers? Ans. 2, 6, 18, 54, and 162.

50. Given $x+y+z+v+w=12.15=a$,

$x+y+z+vw=9.15=b$,

$x+y+zvw=5.68=c$,

$x+yzvw=5.09=d$,

$xyzvw=.45=e$. Required the values of x, y, z, v, and w?

Ans. $x=\dfrac{d\pm\sqrt{(d^2-4e)}}{2}=5$, or $.09$.

$$y=\frac{c-x\pm\sqrt{\{(c-x)^2-4(d-x)\}}}{2}$$

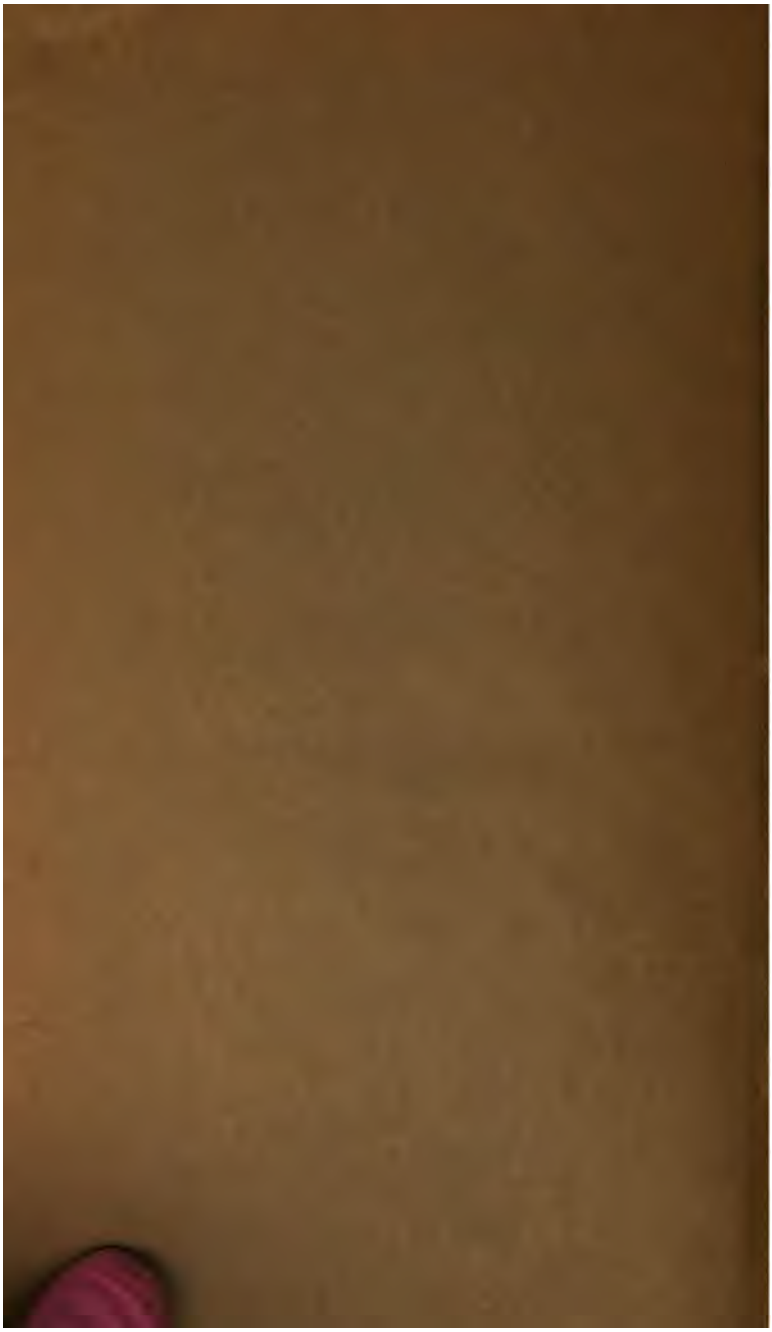
$$z=\frac{b-x-y\pm\sqrt{\{(b-x-y)^2-4(c-x-y)\}}}{2}$$

$$v=\frac{a-x-y-z\pm\sqrt{\{(a-x-y-z)^2-4(b-x-y-z)\}}}{2}$$

$$w=a-x-y-z-v.$$

From the ambiguity of the signs, it is manifest that the number of numerical values of the unknown quantities go on increasing from x to v.

THE END.

ERRATA.

Page 22, example 6, in answer, insert, $-2ab^2c$.
35, line 5, for $+6x$, read $-6x$.
76, line 20, for a^2+, read a^2+b^2.
79, line 14, for 2^{nd}, read 3^{rd}.
105, line 4 from bottom, for b, read c.
105, line 3, read $(1+x)^{n-2}\times\overline{1+y}.....(1+y)^{n-1}$.
121, line 4 from bottom, for (102) read (101.)
150, line 15, read $c'=(ar+b)r+(2ar+b)r+c$.
150, lines 5 and 6 from bottom, read $c''=(as+b')s+(2as+b')s+c$.
160, line 19, for 290, read 270.
161, line 23, for $6\surd5$, read $4\surd5$.
164, line 4, for $\frac{3}{4}$, read $\frac{3}{2}$.
166, line 13, for 2 read 32.